MATH 2403, Spring 2013

Final Practice: Solutions

Problem 1 Newton's law of cooling says that the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$. A 4-lb roast, initially at 50 F, is placed in a 375 F oven at 5:00 pm. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125 F. When will the roast be 150 F?

Solutions: Let k be the proportionality constant, we have the following problem:

$$\begin{cases} \frac{dT}{dt} = k(375 - T), \\ T(0) = 50, \ T(75) = 125. \end{cases}$$

Solve the equation using separation of variables, we have

$$T(t) = 375 - Ce^{-kt}.$$

By $T(0) = 50$, we have $C = 325$. Solving $T(75) = 125$ we further obtain

$$k = \frac{1}{75} \ln(13/10).$$

We now solve for time t_* such that $T(t_*) = 150$. This leads to

$$t_* = 75\frac{\ln(13/9)}{\ln(13/10)},$$

which is about 105 minutes. So, the time is about 6:45 pm.

Problem 2. Solve the following problems.

(a) $y''' + 4y' = 0$, $y(0) = 0$, $y'(0) = 1$ and $y''(0) = 0$.

Solution: $r^3 + 4r = 0$, and so $r_1 = 0$, $r_{2,3} = \pm 2i$. The general solution is

$$y(x) = c_1 + c_2 cos(2x) + c_3 sin(2x).$$

The initial conditions tell $c_1 = 0$, $c_2 = 0$ and $c_3 = \frac{1}{2}$. Hence

$$y(x) = \frac{1}{2} sin(2x).$$

(b) $y'' + 3y' + 2y = 3x$.

Solution: $r^2+3r+2 = 0$, and so $r_1 = -2$, and $r_2 = -1$. The complementary solution is

$$y_c(x) = c_1 e^{-2x} + c_2 e^{-x}.$$

By undetermined coefficient method, we have $y_p(x) = A + Bx$. Substitute it into equation, we have $A = -\frac{9}{4}$, $B = \frac{3}{2}$. Therefore,

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{3}{2}x - \frac{9}{4}.$$

(c) $y'' + 4y' + 4y = 2e^{-2x}$.

Solution: $r^2 + 4r + 4 = 0$ and $r_1 = r_2 = -2$. The complementary solution is

$$y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

By undetermined coefficient method, we have $y_p(x) = Ax^2 e^{-2x}$. Substitute it into equation, we have $A = 1$. Therefore,

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 e^{-2x}.$$

Problem 3. Consider the differential equation $4t^2 y'' - ty' + y = 0$ (which we don't know how to solve).

(a) Show that $y_1(t) = t^{\frac{1}{4}}$ and $y_2(t) = t$ are solutions to this differential eqution (for $t > 0$).

Solution: Substititute them into the equation and check, we found yes.

(b) Write the general solution to the differential equation. What is true about y_1 and y_2 that allows you to do this? How do you know?

Solution: We compute the Wranskian and find

$$W(y_1, y_2) = \frac{3}{4} t^{\frac{1}{4}} > 0, \ for \ t > 0.$$

Therefore, y_1 and y_2 are linearly independent and the general solution is

$$y(t) = c_1 t^{\frac{1}{4}} + c_2 t.$$

(c) Find the particular solution for the differential equatin if $y(1) = 3$ and $y'(1) = 2$.

Solution: Using the general solution from part (b), the initial condition tells $c_1 = \frac{4}{3}$ and $c_2 = \frac{5}{3}$, and therefore

$$y(t) = \frac{4}{3} t^{\frac{1}{4}} + \frac{5}{3} t.$$

(d) Find the general solution to the equation $4t^2 y'' - ty' + y = 5t$

Solution: Chage the equation into standard form

$$y'' - \frac{1}{4t} y' + \frac{1}{4t^2} y = \frac{5}{4t}$$

Set $g(t) = \frac{5}{4t}$. From part (b) and the variation of parameter method, one can find a particular solution of the equation

$$\begin{aligned} y_p(t) &= -y_1(t) \int \frac{y_2(t) g(t)}{W(y_1, y_2)}\, dt + y_2(t) \int \frac{y_1(t) g(t)}{W(y_1, y_2)}\, dt \\ &= -\frac{20}{9} t + \frac{5}{3} t \ln(t). \end{aligned}$$

Therefore, the general solution is

$$y(t) = c_1 t^{\frac{1}{4}} + c_2 t - \frac{20}{9} t + \frac{5}{3} t \ln(t).$$

Problem 4 Consider the following system

$$\begin{cases} \dfrac{dx}{dt} = xy - 1 \\ \dfrac{dy}{dt} = x - y \end{cases}$$

(a) Find all critical points for this system.

Solutions: $P_1(1,1)$ and $P_2(-1,-1)$.

(b) There is a critical point in 3rd quadrant, find the corresponding linearization of your system near this critical point.

Solutions: We compute $J(-1,-1)$,

$$J(-1,-1) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Now, we define $u = x + 1$ and $v = y + 1$, the linearized system is

$$\begin{cases} \frac{du}{dt} = -u - v, \\ \frac{dv}{dt} = u - v, \end{cases}$$

(c) Find the general solution of this linear system.

Solutions: Compute the eigenvalues, one finds $\lambda_{1,2} = -1 \pm i$. Compute the eigenvector of $-1+i$, we find $\mathbf{v} = (i,1)^t = (0,1)^t + i(1,0)^t$. Thus $\mathbf{a} = (0,1)^t$ and $\mathbf{b} = (1,0)$. The general solution is

$$\mathbf{x}(t) = c_1 e^{-t}[\mathbf{a} cost - \mathbf{b} sint] + c_2 e^{-t}[\mathbf{a} sint + \mathbf{b} cost].$$

(d) Sketch the phase portrait for the linear system you found in part (b), indicating the type and the stability of the critical point for this linear system.

Solutions This critital point $(-1,-1)$ on xy-plane or $(0,0)$ on uv-plane, is an asymptotically stable spiral point. The trajectory spiral into the critical point and in the direction of counterclockwise.

Problem 5 Consider the system of equations

$$\begin{cases} \frac{dx}{dt} = -x - 4y, \\ \frac{dy}{dt} = x - y, \end{cases} \tag{1}$$

(a) Find the general solution of (1).

Solution: Coumpute the eigenvalues, we have $\lambda_{1,2} = -1 \pm 2i.$. Compute the eigenvector of $-1+2i$, and found one of them is $\mathbf{v} = (2i, 1)^t = (0, 1)^t + i(2, 0)^t$. Thus, $\mathbf{a} = (0, 1)^t$ and $\mathbf{b} = (2, 0)^t$. Therefore, the general solution is

$$\mathbf{x(t)} = c_1 e^{-t}[\mathbf{a}cos(2t) - \mathbf{b}sin(2t)] + c_2 e^{-t}[\mathbf{a}sin(2t) + \mathbf{b}cos(2t)].$$

(b) Find the particular solution for system (1) with the initial condition $x(0) = 2$ and $y(0) = -1$.

Solution: By the initial condition, we found $c_1 = -1$ and $c_2 = 1$.

(c) Sketch the the solution curves in (b) on the phase plane. Indicate the direction for t increase and the limiting behavior of the solution as t goes to $+\infty$.

Solution: The solution curve in (b) starts at $(2, -1)$ with velocity $(2, 3)$, spiral around the origin counterclockwisely and approaching to the origin as time goes to $+\infty$.

Problem 6 (a) Find the Laplace transform of $f(t) = 4\delta(t-2) + t^3 e^{-3t} + g(t)$ where $g(t) = t$ if $t < 3$ and $g(t) = 3$ if $t > 3$.

Solution: We have the expression of $g(t)$ as

$$g(t) = t(1 - u(t-3)) + 3u(t-3) = t - (t-3)u(t-3).$$

Therefore, using linearity and proper formulas, we have

$$L(f(t)) = 4e^{-2s} + \frac{6}{(s+3)^4} + \frac{1}{s^2} - \frac{e^{-3s}}{s^2}.$$

(b) Use Laplace transform to solve the initial value problem

$$x'' + 3x' + 2x = t + 1;\ x(0) = 0,\ x'(0) = 2.$$

Solution: Apply Laplace tranform both side, using the initial conditions, with $X(s) = L(x(t))$, we have

$$(s^2 + 3s + 2)X(s) = \frac{2s^2 + s + 1}{s^2}.$$

Solve it for $X(s)$, we have

$$X(s) = \frac{2s^2 + s + 1}{s^2(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+1} + \frac{d}{s+2}.$$

By common denomenator and comparint the numerators, we have

$$b = \frac{1}{2},\ a = -\frac{1}{4},\ c = 2,\ d = -\frac{7}{4}.$$

Apply the inverse Laplace transfor, we have

$$x(t) = -\frac{1}{4} + \frac{1}{2}t + 2e^{-t} - \frac{7}{4}e^{-2t}.$$

Problem 7 The charge on a capacitor, $Q(t)$, in a simple LRC circuit is given by $LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$, where L, R, and C are the inductance, resistance and capacitance of the inductor, resistor and capacitor in the system. Suppose that $R = 1k\Omega$, $C = 20\mu F$, $L = 0.35h$, and that the system is forced with an applied charge $E(t) = Asin(\omega t)$.

(a) For what ω, if any, will this system exhibit resonance? Explain.

Solution:

$R = 1 \neq 0$, there is no pure resonance. Furthermore, $\frac{1}{CL} < R^2$, there is no practical resonance either. However, for the current $I = Q'$, the electrical resonance occurs when $\omega = \omega_m = \frac{1}{\sqrt{LC}}$.

(b) Now suppose $R = 0$ (the resistor is removed). If $\omega = 120\pi$ and $A = 117$, find the charge $Q(t)$ if $Q(0) = Q'(0) = 0$. Be sure that it is clear why you proceed as you do and how you arrive at your answer. Will your solution exhibit resonance, beats or neither? Why?

Solution: We solve $LQ''(t) + \frac{1}{C}Q(t) = Asin(\omega t)$ with $Q(0) = Q'(0) = 0$. Let $\omega_0 = \frac{1}{\sqrt{LC}}$, we see it is far from ω, therefore, there is no resonance or beats. The complimentary solution is

$$Q_c(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t).$$

Assume $Q_p(t) = Bsin(\omega t)$, we find

$$B = \frac{A}{(\frac{1}{C} - L\omega^2)}.$$

Therefore,

$$Q(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t) + \frac{A}{(\frac{1}{C} - L\omega^2)} sin(\omega t).$$

By initial condition, we find $c_1 = 0$ and $c_2 = -\frac{\omega}{\omega_0} B$, and so

$$Q(t) = \frac{A}{(\frac{1}{C} - L\omega^2)} (sin(\omega t) - \frac{\omega}{\omega_0} sin(\omega_0 t)).$$