3.3 Problems

Find the general solutions of the differential equations in Problems 1 through 20.

1. $y'' - 4y = 0$
2. $2y'' - 3y' = 0$
3. $y'' + 3y' - 10y = 0$
4. $2y'' - 7y' + 3y = 0$
5. $y'' + 6y' + 9y = 0$
6. $y'' + 5y' + 5y = 0$
7. $4y'' - 12y' + 9y = 0$
8. $y'' - 6y' + 13y = 0$
9. $y'' + 8y' + 25y = 0$
10. $5y^{(4)} + 3y^{(3)} = 0$
11. $y^{(4)} - 8y^{(3)} + 16y'' = 0$
12. $y^{(4)} - 3y^{(3)} + 3y'' - y' = 0$
13. $9y^{(3)} + 12y'' + 4y' = 0$
14. $y^{(4)} + 3y'' - 4y = 0$
15. $y^{(4)} - 8y'' + 16y = 0$
16. $y^{(4)} + 18y'' + 81y = 0$
17. $6y^{(4)} + 11y'' + 4y = 0$
18. $y^{(4)} = 16y$
19. $y^{(3)} + y'' - y' - y = 0$
20. $y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$ (*Suggestion*: Expand $(r^2 + r + 1)^2$.)

Solve the initial value problems given in Problems 21 through 26.

21. $y'' - 4y' + 3y = 0$; $y(0) = 7$, $y'(0) = 11$
22. $9y'' + 6y' + 4y = 0$; $y(0) = 3$, $y'(0) = 4$
23. $y'' - 6y' + 25y = 0$; $y(0) = 3$, $y'(0) = 1$
24. $2y^{(3)} - 3y'' - 2y' = 0$; $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$
25. $3y^{(3)} + 2y'' = 0$; $y(0) = -1$, $y'(0) = 0$, $y''(0) = 1$
26. $y^{(3)} + 10y'' + 25y' = 0$; $y(0) = 3$, $y'(0) = 4$, $y''(0) = 5$

Find general solutions of the equations in Problems 27 through 32. First find a small integral root of the characteristic equation by inspection; then factor by division.

27. $y^{(3)} + 3y'' - 4y = 0$
28. $2y^{(3)} - y'' - 5y' - 2y = 0$
29. $y^{(3)} + 27y = 0$
30. $y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$
31. $y^{(3)} + 3y'' + 4y' - 8y = 0$
32. $y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0$

In Problems 33 through 36, one solution of the differential equation is given. Find the general solution.

33. $y^{(3)} + 3y'' - 54y = 0$; $y = e^{3x}$
34. $3y^{(3)} - 2y'' + 12y' - 8y = 0$; $y = e^{2x/3}$
35. $6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4y = 0$; $y = \cos 2x$
36. $9y^{(3)} + 11y'' + 4y' - 14y = 0$; $y = e^{-x}\sin x$
37. Find a function $y(x)$ such that $y^{(4)}(x) = y^{(3)}(x)$ for all x and $y(0) = 18$, $y'(0) = 12$, $y''(0) = 13$, and $y^{(3)}(0) = 7$.
38. Solve the initial value problem

$$y^{(3)} - 5y'' + 100y' - 500y = 0;$$
$$y(0) = 0, \quad y'(0) = 10, \quad y''(0) = 250$$

given that $y_1(x) = e^{5x}$ is one particular solution of the differential equation.

In Problems 39 through 42, find a linear homogeneous constant-coefficient equation with the given general solution.

39. $y(x) = (A + Bx + Cx^2)e^{2x}$
40. $y(x) = Ae^{2x} + B\cos 2x + C\sin 2x$
41. $y(x) = A\cos 2x + B\sin 2x + C\cosh 2x + D\sinh 2x$
42. $y(x) = (A + Bx + Cx^2)\cos 2x + (D + Ex + Fx^2)\sin 2x$

Problems 43 through 47 pertain to the solution of differential equations with complex coefficients.

43. **(a)** Use Euler's formula to show that every complex number can be written in the form $re^{i\theta}$, where $r \geqq 0$ and $-\pi < \theta \leqq \pi$. **(b)** Express the numbers 4, -2, $3i$, $1+i$, and $-1+i\sqrt{3}$ in the form $re^{i\theta}$. **(c)** The two square roots of $re^{i\theta}$ are $\pm e^{i\theta/2}\sqrt{r}$. Find the square roots of the numbers $2 - 2i\sqrt{3}$ and $-2 + 2i\sqrt{3}$.
44. Use the quadratic formula to solve the following equations. Note in each case that the roots are not complex conjugates.

(a) $x^2 + ix + 2 = 0$ **(b)** $x^2 - 2ix + 3 = 0$

45. Find a general solution of $y'' - 2iy' + 3y = 0$.
46. Find a general solution of $y'' - iy' + 6y = 0$
47. Find a general solution of $y'' = \left(-2 + 2i\sqrt{3}\right)y$.
48. Solve the initial value problem

$$y^{(3)} = y; \quad y(0) = 1, \quad y'(0) = y''(0) = 0.$$

(*Suggestion*: Impose the given initial conditions on the general solution

$$y(x) = Ae^x + Be^{\alpha x} + Ce^{\beta x},$$

where α and β are the complex conjugate roots of $r^3 - 1 = 0$, to discover that

$$y(x) = \frac{1}{3}\left(e^x + 2e^{-x/2}\cos\frac{x\sqrt{3}}{2}\right)$$

is a solution.)

49. Solve the initial value problem

$$y^{(4)} = y^{(3)} + y'' + y' + 2y;$$
$$y(0) = y'(0) = y''(0) = 0, \; y^{(3)}(0) = 30.$$

50. The differential equation

$$y'' + (\operatorname{sgn} x)y = 0 \qquad (25)$$

has the discontinuous coefficient function

$$\operatorname{sgn} x = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Show that Eq. (25) nevertheless has two linearly independent solutions $y_1(x)$ and $y_2(x)$ defined for all x such that

- Each satisfies Eq. (25) at each point $x \neq 0$;
- Each has a continuous derivative at $x = 0$;
- $y_1(0) = y_2'(0) = 1$ and $y_2(0) = y_1'(0) = 0$.

We easily solve these equations for

$$u_1' = -\sin x \tan x = -\frac{\sin^2 x}{\cos x} = \cos x - \sec x,$$
$$u_2' = \cos x \tan x = \sin x.$$

Hence we take

$$u_1 = \int (\cos x - \sec x)\,dx = \sin x - \ln|\sec x + \tan x|$$

and

$$u_2 = \int \sin x\,dx = -\cos x.$$

(Do you see why we choose the constants of integration to be zero?) Thus our particular solution is

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\ &= (\sin x - \ln|\sec x + \tan x|)\cos x + (-\cos x)(\sin x); \end{aligned}$$

that is,

$$y_p(x) = -(\cos x)\ln|\sec x + \tan x|. \qquad \blacksquare$$

3.5 Problems

In Problems 1 through 20, find a particular solution y_p of the given equation. In all these problems, primes denote derivatives with respect to x.

1. $y'' + 16y = e^{3x}$
2. $y'' - y' - 2y = 3x + 4$
3. $y'' - y' - 6y = 2\sin 3x$
4. $4y'' + 4y' + y = 3xe^x$
5. $y'' + y' + y = \sin^2 x$
6. $2y'' + 4y' + 7y = x^2$
7. $y'' - 4y = \sinh x$
8. $y'' - 4y = \cosh 2x$
9. $y'' + 2y' - 3y = 1 + xe^x$
10. $y'' + 9y = 2\cos 3x + 3\sin 3x$
11. $y^{(3)} + 4y' = 3x - 1$
12. $y^{(3)} + y' = 2 - \sin x$
13. $y'' + 2y' + 5y = e^x \sin x$
14. $y^{(4)} - 2y'' + y = xe^x$
15. $y^{(5)} + 5y^{(4)} - y = 17$
16. $y'' + 9y = 2x^2e^{3x} + 5$
17. $y'' + y = \sin x + x\cos x$
18. $y^{(4)} - 5y'' + 4y = e^x - xe^{2x}$
19. $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$
20. $y^{(3)} - y = e^x + 7$

In Problems 21 through 30, set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.

21. $y'' - 2y' + 2y = e^x \sin x$
22. $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$
23. $y'' + 4y = 3x\cos 2x$
24. $y^{(3)} - y'' - 12y' = x - 2xe^{-3x}$
25. $y'' + 3y' + 2y = x(e^{-x} - e^{-2x})$
26. $y'' - 6y' + 13y = xe^{3x}\sin 2x$
27. $y^{(4)} + 5y'' + 4y = \sin x + \cos 2x$
28. $y^{(4)} + 9y'' = (x^2 + 1)\sin 3x$
29. $(D-1)^3(D^2-4)y = xe^x + e^{2x} + e^{-2x}$
30. $y^{(4)} - 2y'' + y = x^2\cos x$

Solve the initial value problems in Problems 31 through 40.

31. $y'' + 4y = 2x$; $y(0) = 1$, $y'(0) = 2$
32. $y'' + 3y' + 2y = e^x$; $y(0) = 0$, $y'(0) = 3$
33. $y'' + 9y = \sin 2x$; $y(0) = 1$, $y'(0) = 0$
34. $y'' + y = \cos x$; $y(0) = 1$, $y'(0) = -1$
35. $y'' - 2y' + 2y = x + 1$; $y(0) = 3$, $y'(0) = 0$
36. $y^{(4)} - 4y'' = x^2$; $y(0) = y'(0) = 1$, $y''(0) = y^{(3)}(0) = -1$
37. $y^{(3)} - 2y'' + y' = 1 + xe^x$; $y(0) = y'(0) = 0$, $y''(0) = 1$
38. $y'' + 2y' + 2y = \sin 3x$; $y(0) = 2$, $y'(0) = 0$
39. $y^{(3)} + y'' = x + e^{-x}$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$
40. $y^{(4)} - y = 5$; $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$

41. Find a particular solution of the equation

$$y^{(4)} - y^{(3)} - y'' - y' - 2y = 8x^5.$$

42. Find the solution of the initial value problem consisting of the differential equation of Problem 41 and the initial conditions

$$y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

43. **(a)** Write

$$\cos 3x + i \sin 3x = e^{3ix} = (\cos x + i \sin x)^3$$

by Euler's formula, expand, and equate real and imaginary parts to derive the identities

$$\cos^3 x = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x,$$
$$\sin^3 x = \tfrac{3}{4}\sin x - \tfrac{1}{4}\sin 3x.$$

(b) Use the result of part (a) to find a general solution of

$$y'' + 4y = \cos^3 x.$$

Use trigonometric identities to find general solutions of the equations in Problems 44 through 46.

44. $y'' + y' + y = \sin x \sin 3x$

45. $y'' + 9y = \sin^4 x$

46. $y'' + y = x\cos^3 x$

In Problems 47 through 56, use the method of variation of parameters to find a particular solution of the given differential equation.

47. $y'' + 3y' + 2y = 4e^x$

48. $y'' - 2y' - 8y = 3e^{-2x}$

49. $y'' - 4y' + 4y = 2e^{2x}$

50. $y'' - 4y = \sinh 2x$

51. $y'' + 4y = \cos 3x$

52. $y'' + 9y = \sin 3x$

53. $y'' + 9y = 2\sec 3x$

54. $y'' + y = \csc^2 x$

55. $y'' + 4y = \sin^2 x$

56. $y'' - 4y = xe^x$

57. You can verify by substitution that $y_c = c_1 x + c_2 x^{-1}$ is a complementary function for the nonhomogeneous second-order equation

$$x^2 y'' + xy' - y = 72x^5.$$

But before applying the method of variation of parameters, you must first divide this equation by its leading coefficient x^2 to rewrite it in the standard form

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 72x^3.$$

Thus $f(x) = 72x^3$ in Eq. (22). Now proceed to solve the equations in (31) and thereby derive the particular solution $y_p = 3x^5$.

In Problems 58 through 62, a nonhomogeneous second-order linear equation and a complementary function y_c are given. Apply the method of Problem 57 to find a particular solution of the equation.

58. $x^2y'' - 4xy' + 6y = x^3$; $y_c = c_1x^2 + c_2x^3$

59. $x^2y'' - 3xy' + 4y = x^4$; $y_c = x^2(c_1 + c_2 \ln x)$

60. $4x^2y'' - 4xy' + 3y = 8x^{4/3}$; $y_c = c_1x + c_2x^{3/4}$

61. $x^2y'' + xy' + y = \ln x$; $y_c = c_1\cos(\ln x) + c_2\sin(\ln x)$

62. $(x^2 - 1)y'' - 2xy' + 2y = x^2 - 1$; $y_c = c_1x + c_2(1 + x^2)$

63. Carry out the solution process indicated in the text to derive the variation of parameters formula in (33) from Eqs. (31) and (32).

64. Apply the variation of parameters formula in (33) to find the particular solution $y_p(x) = -x\cos x$ of the nonhomogeneous equation $y'' + y = 2\sin x$.

3.5 Application: Automated Variation of Parameters

The variation of parameters formula in (33) is especially amenable to implementation in a computer algebra system when the indicated integrals would be too tedious or inconvenient for manual evaluation. For example, suppose that we want to find a particular solution of the nonhomogeneous equation

$$y'' + y = \tan x$$

of Example 11, with complementary function $y_c(x) = c_1\cos x + c_2\sin x$. Then the *Maple* commands

```
y1 := cos(x):
y2 := sin(x):
f := tan(x):
W := y1*diff(y2,x) - y2*diff(y1,x):
W := simplify(W):
yp := -y1*int(y2*f/W,x) + y2*int(y1*f/W,x):
simplify(yp);
```

implement (33) and produce the result

$$y_p(x) = -(\cos x)\ln\left(\frac{1 + \sin x}{\cos x}\right)$$