Math 4305, Spring 2016,
Midterm 1, Practice solutions

Problem 1 Suppose that the matrix below is the augmented matix of a system of linear equations

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & k & h \end{pmatrix}$$

a) For what values of h and k, this system has no solution.

Solution: By interchanging r_2 and r_3, and $r_4 - r_3$, one arrives the REF of the matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & k-3 & h-2 \end{pmatrix}$$

Now, it's easy to see that, the system has no solution if and only if the rightmost column is pivot. This happens if and only if $k = 3$ and $h \neq 2$.

b) For what values of h and k, this system has a unique solution. Find the solution.

Solution: Based on the REF derived from part a), the system has a unique solution if and only if $k \neq 3$. In this case, the system is consistent without free variable. In order to solve the system, we row reduce the REF into RREF. This is achieved by $\frac{1}{k-3}r_4$, $-\frac{1}{2}r_2$, $r_3 - 3r_4$, $r_2 + \frac{1}{2}r_4$ and $r_1 - 2r_2$. The RREF is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4-Y \\ 0 & 1 & 0 & 0 & -\frac{3}{2}+\frac{1}{2}Y \\ 0 & 0 & 1 & 0 & 2-3Y \\ 0 & 0 & 0 & 1 & Y \end{array}\right),$$

where $Y = \frac{h-2}{k-3}$. So the solution is
$x_1 = 4 - Y$, $x_2 = -\frac{3}{2} + \frac{1}{2}Y$, $x_3 = 2 - 3Y$ and $x_4 = Y$.

c) For what values of h and k, this system has infinitely many solutions. Describe the set of all solutions using parametric vector form.

Solution: From part a), the system has infinitely many solutions if and only if $k = 3$ and $h = 2$. In this case, the system is consistent with a free variable x_4. The REF is now

$$\left(\begin{array}{ccccc} 1 & 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

To solve the system, we row reduce the above matrix to RREF by $r_1 + r_2$ and $-\frac{1}{2}r_2$:

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, $x_1 = 4 - x_4$, $x_2 = -\frac{3}{2} + \frac{1}{2}x_4$, $x_3 = 2 - 3x_4$ and x_4 is free. So the solution is described by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{3}{2} \\ 2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ \frac{1}{2} \\ -3 \\ 1 \end{pmatrix}.$$

Problem 2 Let $\mathbf{v} = (1, 0, 1)^t$. Define the linear transformation $T : \mathbf{R^3} \to \mathbf{R^3}$ by $T(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$. Where $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

a) Find the standard matrix A of T.

Solution $A = [\mathbf{a}_1, \mathbf{a_2}, \mathbf{a}_3]$, where $\mathbf{a}_i = T(\mathbf{e}_i)$.

$$T(\mathbf{e}_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ T(\mathbf{e}_2) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ T(\mathbf{e}_3) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

We thus have

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

b) Find a basis of $im(A)$.

Solution: We do the interchange of r_1 and r_2, then $r_3 + r_2$, we thus reach the REF of A:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we know that a basis of $im(A)$ is $\{\mathbf{a}_1, \mathbf{a}_2\}$.

c) What's the dimension of $ker(A)$?

Solution By the Rank Theorem, we know that

$$dim\ ker(A) = 3 - dim\ im(A) = 1.$$

Problem 3 Consider an $m \times n$ matrix A and an $n \times m$ matrix B (with $n \neq m$) such that $AB = I_m$. Are the columns of B linearly independent? What about columns of A?

Solution: If columns of B are linearly dependent, so are columns of AB, which contradicts to $AB = I_m$.Or, we could show that colums of B are linearly independent directly. To show this, we assume there is a vector $\mathbf{x} \in \mathbf{R}^m$, such that $B\mathbf{x} = 0$. Then we have $\mathbf{x} = I_m\mathbf{x} = AB\mathbf{x} = A0 = 0$.Thus, if $AB = I_m$, then columns of B are linearly independent. Furthermore, we know that $n > m$. Since A is $m \times n$, columns of A are linearly dependent.

Problem 4 Let $S = \{(x, y) : xy \geq 0\}$ be a subset of the plane $\mathbf{R}^2$. Is S a subspace of $\mathbf{R}^2$?

Solution: S is not a subspace of $\mathbf{R}^2$. One can easily verify that it is not close for addition. Choose $\mathbf{v} = (-1, 0)$ and $\mathbf{u} = (0, 1)$, both are in S, however, $\mathbf{v} + \mathbf{u} = (-1, 1)$ is not in S.

Problem 5 For which values of the cosntant k is the following matrix invertible? Find the inverse.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$$

Solution: Row reduce the matrix into REF by $r_2 - r_1$, $r_3 - r_1$, $r_3 - 3r_2$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 0 & k^2 - 3k + 2 \end{pmatrix}.$$

The matrix is invertible if $k^2 - 3k + 2 \neq 0$. Thus, if $k \neq 1$ and $k \neq 2$, the matrix is invertible. For $k \neq 1$ and $k \neq 2$, we denote the nonzero quantity $k^2 - 3k + 2$ by N, let $M = k - 1$, thus Gauss-Jordan algorithm will give the inverse

$$\frac{1}{N} \begin{pmatrix} 2M + 2N - 2 & 3 - 3M - N & M - 1 \\ -N - 2M & N + 3M & -M \\ 2 & -3 & 1 \end{pmatrix}.$$