MATH 2401, Summer 2014
Practice Midterm, Solutions

Problem 1. Calculations.

(a) $\frac{d}{dt}[(2t\mathbf{i} + \sqrt{t}\mathbf{j}) \bullet (t\mathbf{i} - 3\mathbf{j})]$

solution: $4t - \frac{3}{2}t^{-1/2}$.

(b) $\frac{d}{dt}[(cost\mathbf{i} + sint\mathbf{j} + t\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})]$

solution: $(5cost - 4)\mathbf{i} + (5sint + 3)\mathbf{j} - (4sint + 3cost)\mathbf{k}$

(c) $\frac{d}{dt}[e^{cos2t}\mathbf{i} + ln(1 + t^2)\mathbf{j} + (1 - cost)\mathbf{k}]$

solution: $-2sin(2t)e^{cos2t}\mathbf{i} + \frac{2t}{1+t^2}\mathbf{j} + sint\mathbf{k}$

(d) Let $w = f(x, y, z) = xz + e^{y^2 z} + \sqrt{xy^2z^3}$, calculate w_x, w_y, w_z, w_{xy} and w_{yz}

solution: $w_x = z + \frac{1}{2}\sqrt{y^2z^3}x^{-1/2}$, $w_y = 2yze^{y^2z} + \sqrt{xz^3}$,

$w_z = x + y^2e^{y^2z} + \frac{3}{2}\sqrt{xy^2z}$, $w_{xy} = \frac{1}{2}z^{3/2}x^{-1/2}$.

$w_{yz} = 2ye^{y^2z} + 2y^3ze^{y^2z} + \frac{3}{2}\sqrt{xz}$.

(e) Set $f(x,y) = \frac{x^2-y^4}{x^3-y^4}$. Determine whether or not f has a limit at $(1,1)$.

solution: Along $x = 1$, the limit is 1, while along $y = 1$, the limit is 2/3. So it has no limit at $(1,1)$.

(f) Find the directional derivative of $f(x,y,z) = xy + yz + zx$ at $P(1,-1,1)$ in the direction of $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Solution:

$$\nabla f = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (y+x)\mathbf{k},$$

$\nabla f(1,-1,1) = 2\mathbf{j}$. $\mathbf{u} = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, so

$$f'_{\mathbf{u}}(1,-1,1) = \nabla f(1,-1,1) \bullet \mathbf{u} = \frac{2}{3}\sqrt{6}.$$

(g) Find the rate of change of $f(x,y) = xe^y + ye^{-x}$ along the curve $\mathbf{r}(t) = (lnt)\mathbf{i} + t(lnt)\mathbf{j}$.

Solution:

$$\nabla f = (e^y - ye^{-x})\mathbf{i} + (xe^y + e^{-x})\mathbf{j},$$

$$\nabla f(\mathbf{r}(t)) = (t^t - lnt)\mathbf{i} + (t^t lnt + \frac{1}{t})\mathbf{j},$$

$$\frac{df}{dt} = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = t^t(\frac{1}{t} + lnt + (lnt)^2) + \frac{1}{t}.$$

(h) Find $\frac{\partial u}{\partial s}$ for $u = x^2 - xy$, $x = scost$, $y = tsins$.

Solution:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (2x-y)(cost) + (-x)(tcoss) = 2scos^2t - t\ sins\ cost - st\ coss\ cost.$$

(i) Find $\frac{dy}{dx}$ if $xcos(xy) + ycos(x) = 2$.

Solution: Set $u = xcos(xy) + ycos(x) - 2$,

$$\frac{\partial u}{\partial x} = cos(xy) - xysin(xy) - ysin(x).$$

$$\frac{\partial u}{\partial y} = -x^2sin(xy) + cos(x).$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{cos(xy) - xysin(xy) - ysin(x)}{x^2sin(xy) - cos(x)}.$$

(j) Is $\mathbf{F}(x, y) = (x + siny)\mathbf{i} + (xcosy - 2y)\mathbf{j}$ a gradient of a function $f(x, y)$? If yes, find the general form of $f(x, y)$.

Solution: Set $P = x + siny$, $Q = xcosy - 2y$. $\frac{\partial P}{\partial y} = cosy = \frac{\partial Q}{\partial x} = cosy$.

Thus, $\mathbf{F}$ is a gradient of a function. For $f(x, y)$, we have from $\frac{\partial f}{\partial x} = P$ that

$$f(x, y) = \frac{1}{2}x^2 + xsin(y) + g(y).$$

To determine $g(y)$, we have

$$Q = \frac{\partial f}{\partial y} = xcos(y) + g'(y),$$

which implies that $g'(y) = -2y$, thus $g(y) = -y^2 + C$, with C a constant. So, $f(x,y) = \frac{1}{2}x^2 + xsin(y) - y^2 + C$.

Problem 2 A golf ball is hit at time $t = 0$. Its position vector as a function of time is given by

$$\mathbf{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} + (-t^2 + 4t)\mathbf{k}.$$

Notice that at $t = 0$ the ball is at the origin of the coordinate system. The xy plane represents the ground. At some time $t_1 > 0$ the ball will return to the xy plane at some point $P(a, b, 0)$.

(a) Compute the velocity, the accelaration and the speed of the ball at an arbitrary time t.

solution: $\mathbf{v}(t) = \mathbf{r}'(t) = 2\mathbf{i} + 3\mathbf{j} + (4 - 2t)\mathbf{k}$, $\mathbf{a}(t) = -2\mathbf{k}$. $v(t) = \|\mathbf{v}(t)\| =$

$\sqrt{13 + (4 - 2t)^2}$.

(b) Find the time $t_1 > 0$ and the coordinates of the point P where the ball hits the xy plane again.

solution: $t_1 = 4$.

(c) Set up a definite integral equal to the length of the arc of the trajectory from the origin to the point P. You do not have to evaluate the integral.

solution: $\int_0^4 \sqrt{13 + (4 - 2t)^2}\, dt.$

(d) Find the equation of the line tangent to the trajectory at P.

solution: $\mathbf{R}(u) = 8\mathbf{i} + 12\mathbf{j} + u\mathbf{r}'(4) = (8 + 2u)\mathbf{i} + (12 + 3u)\mathbf{j} - 4u\mathbf{k}.$

(e) Find the equation of the vertical plane containing the trajectory.

solution: The plane is through the origin and is vertical. The vector $\mathbf{k}$ is in this plane. The other vector can be easily found by the projection of the trajectory onto xy-plane. The projection of the trajectory onto xy-plane is the curve:

$C(t)$: $x = 2t$, $y = 3t$, $z = 0$.

This is a straight line through the origin with dirction $2\mathbf{i} + 3\mathbf{j}$. Thus the normal for the plane of the trajectory is

$\mathbf{n} = (2\mathbf{i} + 3\mathbf{j}) \times \mathbf{k} = 3\mathbf{i} - 2\mathbf{j}.$

The plane equation is thus

$3x - 2y = 0.$

(f) Find the curvature of the trajectory at P.

solution: (1)$k = \frac{\|\mathbf{v}\times\mathbf{a}\|}{(ds/dt)^3}$. At $t_1 = 4$,

$$k = \frac{\sqrt{52}}{29^{\frac{3}{2}}}.$$

(2) $k = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{\sqrt{52}}{29^{\frac{3}{2}}}$.

Problem 3 At each point $P(x(t), y(t), z(t))$ of its motion, an object of mass m is subject to a force:

$\mathbf{F}(t) = m\pi^2[4cos(\pi t)\mathbf{i} + 3sin(\pi t)\mathbf{j}]$. Given that $\mathbf{v}(0) = -3\pi\mathbf{j} + \mathbf{k}$, and $\mathbf{r}(0) = 3\mathbf{j}$. find the following:

solution: Upon integration, we have

$\mathbf{a}(t) = \pi^2[4cos(\pi t)\mathbf{i} + 3sin(\pi t)\mathbf{j}]$,

$\mathbf{v}(t) = 4\pi sin(\pi t)\mathbf{i} - 3\pi cos(\pi t)\mathbf{j} + \mathbf{k}$.

$\mathbf{r}(t) = 4(1 - cos(\pi t))\mathbf{i} + 3(1 - sin(\pi t)\mathbf{j} + t\mathbf{k}$.

(a) The velocity $\mathbf{v}(1)$.

solution: $\mathbf{v}(1) = 3\pi\mathbf{j} + \mathbf{k}$.

(b) The speed $v(1)$.

solution: $v(1) = \sqrt{9\pi^2 + 1}$.

(c) The momentum $\mathbf{p}(1)$.

solution: $\mathbf{p}(1) = m\mathbf{v}(1) = 3m\pi\mathbf{j} + m\mathbf{k}$.

(d) The angular momentum $\mathbf{L}(1)$.

solution: $\mathbf{L}(1) = \mathbf{r}(1) \times \mathbf{p}(1) = 3m(1-\pi)\mathbf{i} - 8m\mathbf{j} + 24m\pi\mathbf{k}$.

(e) The torque $\tau(1)$.

solution: $\tau(1) = \mathbf{r}(1) \times \mathbf{F}(1) = -4m\pi^2\mathbf{j} + 12m\pi^2\mathbf{k}$.

(f) The position $\mathbf{r}(1)$.

solution: $\mathbf{r}(1) = 8\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Problem 4 Consider the function $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$.

(a) Find the equation for the tangent plane to the level surface $f = 4$ at the point $P(1, 4, 1)$.

Solution:

$$\nabla f = \frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2\sqrt{y}}\mathbf{j} + \frac{1}{2\sqrt{z}}\mathbf{k},$$

$$\nabla f(1, 4, 1) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{1}{2}\mathbf{k}.$$

Tangent plane: $\frac{1}{2}(x-1) + \frac{1}{4}(y-4) + \frac{1}{2}(z-1) = 0$.

(b) Find the equation for the normal line to $f = 4$ at $P(1, 4, 1)$.

Solution: The normal line: $x = 1 + \frac{1}{2}t$, $y = 4 + \frac{1}{4}t$, $z = 1 + \frac{1}{2}t$.

(c) Use differentials to estimate $f(0.9, 4.1, 1.1)$.

Solution: $f(0.9, 4.1, 1.1) \tilde{-} f(1, 4, 1) + df$.

$$df = \frac{1}{2} \times (-0.1) + \frac{1}{4} \times 0.1 + \frac{1}{2} \times 0.1 = 0.025.$$

Thus, the estimate is 4.025.

Problem 5. Find the area of the largest rectangle with edges parrallel to the coordinate axes that can be inscribed in the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Solution: Use Lagrange multiplies method. Set the coordiates of the corner points of the rectangle to be (x, y), $(-x, y)$, $(-x, -y)$, $(x, -y)$. We need to

maximize $f(x, y) = 4xy$ with the side condition $g(x, y) = \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$. $\nabla f = 4y\mathbf{i} + 4x\mathbf{j}$, $\nabla g = \frac{2}{9}x\mathbf{i} + \frac{1}{2}y\mathbf{j}$. Solve the following system:

$$\begin{cases} 4y = \lambda\frac{2}{9}x \\ 4x = \lambda\frac{1}{2}y \\ g(x, y) = 0. \end{cases}$$

We have

$\lambda = 12$, $x = \frac{3}{2}\sqrt{2}$, $y = \sqrt{2}$, and the area is 12.

Problem 6 Find the absolute extreme values taken on $f(x, y) = \frac{-2y}{x^2+y^2+1}$ on

the set $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

Solution: $\nabla f = \frac{4xy}{(x^2+y^2+1)^2}\mathbf{i} + \frac{2y^2-2x^2-2}{(x^2+y^2+1)^2}\mathbf{j} = \mathbf{0}$ at $P_1 = (0,1)$ and $P_2 = (0,-1)$ in D.

Next we consider the boundary of D. We parametrize the circle by

$$C : \mathbf{r}(t) = 2cos(t)\mathbf{i} + 2sin(t)\mathbf{j},\ t \in [0, 2\pi].$$

The values of f on the boundary are given by the function:

$$F(t) = f(\mathbf{r}(t)) = -\frac{4}{5}sin(t),\ t \in [0, 2\pi].$$

$F'(t) = -\frac{4}{5}cos(t) = 0$ at $t = \frac{1}{2}\pi$ and $t = \frac{3}{2}\pi$. Thus the critical points on boundary of D are $P_3 = \mathbf{r}(0) = \mathbf{r}(2\pi) = (2,0)$, $P_4 = \mathbf{r}(\frac{1}{2}\pi) = (0,2)$, and $P_5 = \mathbf{r}(\frac{3}{2}\pi) = (0,-2)$. Evaluate f at all critical points:

$$f(0,1) = -1,\ f(0,-1) = 1,\ f(2,0) = 0,$$

$$f(0,2) = -\frac{4}{5},\ f(0,-2) = \frac{4}{5}.$$

So, f takes on its absolute maximum of 1 at $(0,-1)$ and its absolute minimum of -1 at $(0,1)$.