MATH 2401, Summer 2014
Practice Final : Solutions

Problem 1. This problem is about the function

$$f(x, y, z) = 3zy + 4xcos(z).$$

(a) Find the rate of change of the function f at $(1, 1, 0)$ in the direction from this point to the origin.

Solution: The direction vector is $\mathbf{v} = -\mathbf{i} - \mathbf{j}$. Normalize it one obtains:$\mathbf{u} = -\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Compute the gradient of f at $(1, 1, 0)$, we have

$$\nabla f(1, 1, 0) = 4\mathbf{i} + 3\mathbf{k}.$$

Thus: $f'_{\mathbf{u}}(1, 1, 0) = \nabla f(1, 1, 0) \bullet \mathbf{u} = -\frac{4}{\sqrt{2}}$.

(b) Give an approximate value of $f(0.9, 1.2, 0.11)$

Solution: To approximate $f(0.9, 1.2, 0.11)$, we use differentials. We know that $f(1, 1, 0) = 4$, and $\Delta x = -0.1$, $\Delta y = 0.2$, $\Delta z = 0.11$. Thus,

$$f(0.9, 1.2, 0.11) \approx f(1, 1, 0) + df = 4 + 4(-0.1) + 0(0.2) + 3(0.11) = 3.93.$$

(c) The equation $f(x, y, z) = 4$ implicitly defines z as a function of (x, y), if we agree that $z = 0$ if $(x, y) = (1, 1)$. Find the numnerical values of the derivatives:

$\frac{\partial z}{\partial x}(1,1)$ and $\frac{\partial z}{\partial y}(1,1)$.

Solution: By the implicit differentiation, we have

$$\frac{\partial z}{\partial x}(1,1) = -\frac{\partial f/\partial x(1,1,0)}{\partial f \partial z(1,1,0)} = -\frac{4}{3}$$

$$\frac{\partial z}{\partial y}(1,1) = -\frac{\partial f/\partial y(1,1,0)}{\partial f \partial z(1,1,0)} = -\frac{0}{3} = 0.$$

(d) Suppose $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a parametric curve such that $\mathbf{r}(0) = (1,1,0)$ and $\mathbf{r}'(0) = (3,2,1)$. Find the value of

$$\frac{d}{dt} f(\mathbf{r}(t))|_{t=0}.$$

Solution: By chain rule,

$$\frac{d}{dt} f(\mathbf{r}(t))|_{t=0} = \nabla f(\mathbf{r}(0) \bullet \mathbf{r}'(0) = (4\mathbf{i} + 3\mathbf{k}) \bullet (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 15.$$

Problem 2 (a) Find the value of a such that the field on the plane

$$\mathbf{F}(x,y) = (axy)\mathbf{i} + x^2\mathbf{j}$$

is conservative. Find a potential for the resulting field.

Solution: Set $P = axy$, $Q = x^2$. For $\mathbf{F}(x,y)$ to be conservative, we need

$$\frac{\partial P}{\partial y} = ax = \frac{\partial Q}{\partial x} = 2x.$$

Hence, $a = 2$.

We now look for $f(x,y)$ such that $\nabla f = \mathbf{F}$. To this purpose, we know from $\frac{\partial f}{\partial x} = P = 2xy$ that

$$f(x,y) = x^2y + g(y).$$

However, $\frac{\partial f}{\partial y} = A = x^2 = x^2 + g'(y)$. This implies $g'(y) = 0$. Thus

$$f(x,y) = x^2y + C$$

.

A potential of $\mathbf{F}$ is $G(x,y) = -x^2y$.

(b) Compute the line integral of the conservative field you found in part (a) over the curve $\mathbf{r}(t) = e^{t^2}\mathbf{i} + tcos(2\pi t)\mathbf{j}$, $0 \leq t \leq 1$.

Solution: We first determine the endpoints for the curve. It is clear that the curve starts at $(1,0)$ and ends at $(e,1)$. Since $\mathbf{F} = \nabla f$, by the fundamental theorem of line integrals, we have

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = f(e,1) - f(1,0) = e^2.$$

Problem 3. Evaluate $I = \int_{C_R} dx + x^2 y dy$, where C_R is the triangle with vertices $(0,0)$, $(0,R)$, $(R,0)$ oriented counterclockwise.

Solution: A convenient way is to apply Green's Theorem. Set $P = 1$, $Q = x^2 y$, we have

$$\begin{aligned} \oint_{C_R} dx + x^2 y dy &= \int\int_D 2xydxdy = \int_0^R \int_0^{R-y} 2xy\ dxdy \\ &= \int_0^R y(R-y)^2\ dy = \frac{1}{2}R^4 - \frac{2}{3}R^4 + \frac{1}{4}R^4 \\ &= \frac{R^4}{12}. \end{aligned}$$

Problem 4 Let S be the portion of the surface $x = 5 - y^2 - z^2$ in the half space $x \geq 1$, oriented so that the normal vector at $(5,0,0)$ is equal to $\mathbf{i}$. Let $\mathbf{F}(x,y,z) = -\mathbf{i} + \mathbf{j}$ (a constant vector field).

(a) Set up and evaluate the flux of $\mathbf{F}$ across S.

Solution: Step 1: We first paramatrize the surface S by $\mathbf{r}(y,z) = (5 - y^2 - z^2)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $(y,z) \in D$. Here D is the disc

$$y^2 + z^2 \leq 4.$$

Step 2: We now compute the fundamental vector product $\mathbf{N}(\mathbf{y}, \mathbf{z})$.

$$\mathbf{r}'_y = -2y\mathbf{i} + \mathbf{j},$$

$$\mathbf{r}'_z = -2z\mathbf{i} + \mathbf{k},$$

$$\mathbf{N} = \mathbf{r}'_y \times \mathbf{r}'_z = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

We confirm that $\mathbf{N}(0,0) = \mathbf{i}$. Set $\mathbf{n}$ be unit vector normalized from $\mathbf{N}$.

Step 3: We now compute the flux of $\mathbf{F}$ across S:

$$\begin{aligned}
the\ flux\ &= \int\int_S \mathbf{F} \cdot \mathbf{n}\ d\sigma \\
&= \int\int_D \mathbf{F} \cdot \mathbf{N}\ dydz \\
&= \int\int_D (-1 + 2y)\ dydz \\
&= \int_0^{2\pi} \int_0^2 (-1 + 2rcos\theta)\ rdrd\theta \\
&= \int_0^{2\pi} (-2 + 4cos(\theta))\ d\theta \\
&= -4\pi.
\end{aligned}$$

(b) Verify that $\mathbf{F} = \nabla \times \mathbf{G}$, where $\mathbf{G} = z\mathbf{j} - x\mathbf{k}$.

Solution: Obivious, omitted.

(c) Give an alternative calculation of the surface integral of part (a) by applying Stokes' theorem.

Solution: The bounding curve of C is $y^2 + z^2 = 4$ oriented in the counterclockwise direction coresponding to $\mathbf{i}$. C is parametrized as $y = 2cos\theta$, $z = 2sin\theta$, with $\theta \in [0, 2\pi]$. Along C, $x = 1$.

By Stokes' Theorem, we can compute the flux as following:

$$
\begin{aligned}
the\ flux &= \int\int_S \mathbf{F}\cdot\mathbf{n}\ d\sigma \\
&= \int\int_S (\nabla\times\mathbf{G}\cdot\mathbf{n})\ d\sigma \\
&= \oint_C zdy - xdz \\
&= \int_0^{2\pi} [2sin(\theta)(-2sin(\theta)) - 2cos(\theta)]\ d\theta \\
&= \int_0^{2\pi} (-4sin^2(\theta) - 2cos(\theta))\ d\theta \\
&= -4\pi.
\end{aligned}
$$

Problem 5 For each item, circle the correct answer or indicate if the statement is ture or false. Assume that the functions, fields and curves below are smooth.

Solution:

(a) Let C be an arc from $(0,0)$ to $(2,1)$. According to the fundamental theorem for line integrals, $\int_C (y-1)dx + (x+2y)dy$ is equal to

(1) 2, (2) 1, (3) It depends on what C is.

Solution: The vector field $(y-1)\mathbf{i} + (x+2y)\mathbf{j} = \nabla f$ with $f = xy - x + y^2$, and therefore the fundamental theorem applies. The correct answer is $f(2,1) - f(0,0) = 1$. So, Choose (2).

(b) For every smooth function f, the integral $\int_0^1 \int_0^{2y^2+1} f(x,y)\ dxdy$ is equal to

(1) $\int_0^3 \int_0^{\sqrt{\frac{1}{2}(x-1)}} f(x,y)\ dydx$,

(2) $\int_1^3 \int_0^{\sqrt{\frac{1}{2}(x+1)}} f(x,y)\ dydx$,

(3) None of the above.

Solution: The region of integration is of type II but not of type I. So correct answer is (3).

(c) If $\mathbf{F}$ is a field such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ where C is the unit circle, then $\mathbf{F}$ must be conservative.

(1) True, (2) False.

Solution: One cannot conclude that $\mathbf{F}$ is conservative just by knowing that the integral of $\mathbf{F}$ around a particular closed curve is zero. One would need to know that the integra of $\mathbf{F}$ around every closed curve is zero to conclude that $\mathbf{F}$ is conservative. So the correct answer is (2).

(d) If C is the boundary of a bounded domain D and C is oriented as in the statement of Green's theorem, then $\int_C x^2y dx - y dy$ equals

(1) $\int\int_D (2xy - 1) dxdy$

(2) $\int\int_D (1-x^2)dxdy$

(3) $\int\int_D (-x^2)dxdy$

(4) None of the above.

Solution: In this case, $Q_x - P_y = -x^2$, so the correct answer is (3) by Green's Theorem.

(e) If (a,b) is a critical point of a function f, and if

$$f_{xx}(a,b) = -2, \ and \ f_{yy}(a,b) = 1,$$

then what can one say about (a,b)?

(1) Noting can be concluded from the given information.

(2) (a,b) is a local minimum of f

(3) (a,b) is a local maximum of f

(4) (a,b) is a saddle point of f

Solution: It is tempting to conclude that, since we don't know anything about the value $f_{xy}(a,b)$, the correct answer should be (1). However, the discriminant of this function at (a,b) is

$$-2 \times 1 - (f_{xy}(a,b))^2 \leq -2 < 0,$$

and therefore the correct answer is (4).

Problem 6 Consider the surface S that is the part of the cone $z = \sqrt{x^2+y^2}$

below the plane $z = 3$.

(a) Give a parametric representation of S. Make sure to explicitly describe or sketch the parametrization domain D.

Solution: We can parametrize S by $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (x^2 + y^2)\mathbf{k}$, where $(x, y) \in D$, and D is the disc

$$x^2 + y^2 \leq 9.$$

(b) Find an equation of the tangent plane to S at the point $P(-1, 1, \sqrt{2})$.

Solution: Let $g(x, y, z) = \sqrt{x^2 + y^2} - z$, S is the level surface of $f(x, y, z) = 0$.

$$\nabla g(-1, 1, \sqrt{2}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \mathbf{k}.$$

So the tangent plane to S at $P(-1, 1, \sqrt{2})$ is

$$-\frac{1}{\sqrt{2}}(x + 1) + \frac{1}{\sqrt{2}}(y - 1) - (z - \sqrt{2}) = 0.$$

(c) If the density function $\lambda(x, y, z)$ is equal to the distance to the xy-plane, find the total mass of the surface S.

Solution:$f(x, y) = \sqrt{x^2 + y^2}$, $x^2 + y^2 \leq 9$.

$$\begin{aligned} M &= \int\int_S \lambda(x, y, z)\ d\sigma \\ &= \int\int_D \sqrt{x^2 + y^2}\sqrt{f_x^2 + f_y^2 + 1}\ dxdy \\ &= \int\int_D \sqrt{x^2 + y^2}\sqrt{2}\ dxdy \\ &= \int_0^{2\pi}\int_0^3 \sqrt{2}rr\ drd\theta \\ &= 18\sqrt{2}\pi. \end{aligned}$$

Problem 7 Let E denote the portion of the solid ball of radius R in the first octant, and let

$$\mathbf{F} = (2x + y)\mathbf{i} + y^2\mathbf{j} + cos(xy)\mathbf{k}.$$

Applying the Divergence Theorem, compute the net flux of the field $\mathbf{F}$ across the boundary of E, oriented by the outward-pointing normal vectors.

Solution: The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = 2 + 2y.$$

By the divergence theorem, the flux out of the given suface is equal to

$$\int\int\int_E (2 + 2y)dxdydz = 2(volum(E)) + 2\int\int\int_E ydxdydz,$$

where E is the region inside the surface. The volume of E is one eigth of the volume of the ball of radius R. Thus

$$2(volum(E)) = \frac{1}{3}\pi R^3.$$

In spherical coordinates, we have

$$\begin{aligned}
2\int\int\int_E y dx dy dz &= 2\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \rho sin(\theta)sin(\phi)\rho^2 sin(\phi)\ d\rho d\theta d\phi \\
&= \frac{R^4}{2}\int_0^{\pi/2}\int_0^{\pi/2} sin(\theta)sin^2(\phi)\ d\theta d\phi \\
&= \frac{R^4}{2}\int_0^{\pi/2} sin^2(\phi)\ d\phi \\
&= \frac{R^4}{8}\pi.
\end{aligned}$$

So the final answer is

$$\frac{1}{3}\pi R^3 + \frac{1}{8}\pi R^4.$$

Problem 8 Please complete the course survey. Your comments will help me to improve my teaching in the future. Thank you in advance.