# Math 4305, Summer 2015,

## Practice midterm I: solutions

**Problem 1** Suppose that the matrix below is the augmented matix of a system of linear equations

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & k & h \end{pmatrix}$$

a) For what values of $h$ and $k$, this system has no solution.

**Solution**: By interchanging $r_2$ and $r_3$, and $r_4 - r_3$, one arrives the REF of the matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & k-3 & h-2 \end{pmatrix}$$

Now, it's easy to see that, the system has no solution if and only if the rightmost column is pivot. This happens if and only if $k = 3$ and $h \neq 2$.

b) For what values of $h$ and $k$, this system has a unique solution. Find the solution.

**Solution**: Based on the REF derived from part a), the system has a unique solution if and only if $k \neq 3$. In this case, the system is consistent without free variable. In order to solve the system, we row reduce the REF into RREF. This is achieved by $\frac{1}{k-3}r_4$, $-\frac{1}{2}r_2$, $r_3 - 3r_4$, $r_2 + \frac{1}{2}r_4$ and $r_1 - 2r_2$. The RREF is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 4-Y \\ 0 & 1 & 0 & 0 & -\frac{3}{2}+\frac{1}{2}Y \\ 0 & 0 & 1 & 0 & 2-3Y \\ 0 & 0 & 0 & 1 & Y \end{pmatrix},$$

where $Y = \frac{h-2}{k-3}$. So the solution is
$x_1 = 4 - Y$, $x_2 = -\frac{3}{2} + \frac{1}{2}Y$, $x_3 = 2 - 3Y$ and $x_4 = Y$.

c) For what values of $h$ and $k$, this system has infinitely many solutions. Describe the set of all solutions using parametric vector form.

**Solution**: From part a), the system has infinitely many solutions if and only if $k = 3$ and $h = 2$. In this case, the system is consistent with a free variable $x_4$. The REF is now

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To solve the system, we row reduce the above matrix to RREF by $r_1 + r_2$ and $-\frac{1}{2}r_2$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $x_1 = 4 - x_4$, $x_2 = -\frac{3}{2} + \frac{1}{2}x_4$, $x_3 = 2 - 3x_4$ and $x_4$ is free. So the solution is described by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{3}{2} \\ 2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ \frac{1}{2} \\ -3 \\ 1 \end{pmatrix}.$$

**Problem 2** Let $\mathbf{v} = (1,0,1)^t$. Define the linear transformation $T : \mathbf{R^3} \to \mathbf{R^3}$ by $T(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$. Where $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

a) Find the standard matrix $A$ of $T$.

**Solution** $A = [\mathbf{a}_1, \mathbf{a_2}, \mathbf{a}_3]$, where $\mathbf{a}_i = T(\mathbf{e}_i)$.

$$T(\mathbf{e}_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ T(\mathbf{e}_2) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ T(\mathbf{e}_3) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

We thus have

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

b) Solve $A\mathbf{x} = \mathbf{0}$.

**Solution**: For the augemented matrix $[A0]$, we do the interchange of $r_1$ and $r_2$, then $r_3 + r_2$, and finally $-r_2$, we thus reach the RREF:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There solution set is therefore

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

**Problem 3** For which values of the constant $k$ is the following matrix invertible? Find the inverse.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$$

**Solution**: Row reduce the matrix into REF by $r_2 - r_1$, $r_3 - r_1$, $r_3 - 3r_2$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{pmatrix}.$$

The matrix is invertible if $k^2 - 3k + 2 \neq 0$. Thus, if $k \neq 1,\ or 2$, the matrix is invertible. For $k \neq 1$ and $k \neq 2$, we denote the nonzero quantity $k^2 - 3k + 2$ by $N$, let $M = k - 1$, thus Gauss-Jordan algorithm will give the inverse

$$\frac{1}{N}\begin{pmatrix} 2M+2N-2 & 3-3M-N & M-1 \\ -N-2M & N+3M & -M \\ 2 & -3 & 1 \end{pmatrix}.$$

**Problem 4** Consider an $n \times m$ matrix $A$ of rank $n$. Show that there exists an $m \times n$ matrix $X$ such $AX = I_n$. If $n < m$, how many such matrices $X$ are there?

**Proof**. Since $A$ is $n \times m$ with rank $n$, $A$ has a pivot in each row, and thus $A\mathbf{x} = \mathbf{b}$ has at least a solution for each $\mathbf{b} \in \mathbf{R}^n$. Now, we choose the $j$-th column of $X$ to be any solution of $A\mathbf{x} = \mathbf{e_j}$ for $j = 1, \cdots, n$. Such $X$ will satisfy

$AX = I_n$.

Now, if $n < m$, $A$ must have some non-pivot columns, and $A\mathbf{x} = \mathbf{e_j}$ has infinitely many solutions. Therefore, there are infinitely many such matices $X$ that $AX = I_n$.