Math 4305, Summer 2015,
Practice Exam 2,solutions

Problem 1 Let $\mathbf{v} = (1,0,1)^t$. Define the linear transformation $T : \mathbf{R^3} \to \mathbf{R^3}$ by $T(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$. Where $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

a) Find the standard matrix A of T.

Solution $A = [\mathbf{a}_1, \mathbf{a_2}, \mathbf{a}_3]$, where $\mathbf{a}_i = T(\mathbf{e}_i)$.

$$T(\mathbf{e}_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ T(\mathbf{e}_2) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ T(\mathbf{e}_3) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

We thus have

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

b) Find a basis of $im(A)$.

Solution: We do the interchange of r_1 and r_2, then $r_3 + r_2$, we thus reach the REF of A:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we know that a basis of $im(A)$ is $\{\mathbf{a}_1, \mathbf{a}_2\}$.

c) What's the dimension of $ker(A)$?

Solution By the Rank Theorem, we know that

$$dim\ ker(A) = 3 - dim\ im(A) = 1.$$

Problem 2 Consider an $m \times n$ matrix A and an $n \times m$ matrix B (with $n \neq m$) such that $AB = I_m$. Are the columns of B linearly independent? What about columns of A?

Solution: If columns of B are linearly dependent, so are columns of AB. Thus, if $AB = I_m$, then columns of B are linearly independent. Furthermore, we know that $n > m$ (otherwise, columns of B are linearly dependent). Since A is $m \times n$, columns of A are linearly dependent.

Problem 3 Find a basis $\mathbf{B}$ of $\mathbf{R}^2$ such that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathbf{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathbf{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solution: Let the vectors in $\mathbf{B}$ be $\mathbf{v}_1$ and $\mathbf{v}_2$, and the matrix $S = [\mathbf{v}_1, \mathbf{v}_2]$. For any $\mathbf{x} \in \mathbf{R}^2$, it is clear that $S[\mathbf{x}]_{\mathbf{B}} = \mathbf{x}$. Therefore, one reads from the problem that

$$SA = C$$

where

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

Therefore

$$S = CA^{-1} = \begin{pmatrix} 12 & -7 \\ 14 & -8 \end{pmatrix}.$$

Remark One can also solve in the way like $3v_1 + 5v_2 = (1,2)^t$ and $2v_1 + 3v_2 = (3,4)^t$. This way is simpler in calculations.

Problem 4 Find all possible values of a so that the columns of A given below are linearly dependent?

$$\begin{pmatrix} a & 2a & 0 & 0 \\ 0 & 0 & a-3 & 3(a-3) \\ 0 & -2a & 0 & 1 \\ 0 & 0 & a-2 & 2(a-2) \end{pmatrix}$$

Solution: Clearly, if $a = 0$, first two columns are zero columns, A has linearly dependent columns. Furthermore, if $a = 2$ or $a = 3$, A has zero rows, it will not have 4 pivots and thus A must have linearly dependet columns. Now, assumpe $a \neq 0$, $a \neq 2$ and $a \neq 3$, by sclaing, A becomes

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & -2a & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

which has REF

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -2a & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

There is a pivot in each columns, and thus columns of A are linearly independet.

Columns of A are linearly dependent if and only if $a = 0$, or $a = 2$ or $a = 3$.

Problem 5 (a) Prove that the set $\mathbf{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $\mathbf{P}_2$.

Proof Since $dim\mathbf{P}_2 = 3$ and $\mathbf{B}$ has 3 vectors, it is sufficient to show that $\mathbf{B}$ is linearly independet. Fix the standard basis $S = \{1, t, t^2\}$, we check the corresponding coordinate vectors are linearly independent. Let $p_1(t) = 1+t^2$, $p_2(t) = t + t^2$, and $p_3(t) = 1 + 2t + t^2$, we have

$$A = [[p_1]_S, [p_2]_S, [p_3]_S] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

We easily verify that A is invertible and thus proves that B is linearly independent and so is a basis for $\mathbf{P}_2$.

b) Find the matrix of the linear transformation $T(f(t)) = f' - 3f$ from $\mathbf{P}_2$ to $\mathbf{P}_2$ withe respect to the basis $\mathbf{B}$ found in part (a).

Solution: The desired matrix M can be obtained by

$$AM = [[T(p_1)]_S, [T(p_2)]_S], [T(p_3)]_S] = D.$$

Since $T(p_1) = -3+2t-3t^2$, $T(p_2) = 1-t-3t^2$ and $T(p_3) = -1-4t-3t^2$, thus

$$M = A^{-1}D = A^{-1}\begin{pmatrix} -3 & 1 & -1 \\ 2 & -1 & -4 \\ -3 & -3 & -3 \end{pmatrix} = \begin{pmatrix} -4 & -\frac{1}{2} & 0 \\ 0 & -4 & -2 \\ 1 & \frac{3}{2} & -1 \end{pmatrix}.$$