

# The 3D Compressible Euler Equations With Damping in a Bounded Domain

Ronghua Pan\* and Kun Zhao

School of Mathematics, Georgia Institute of Technology  
Atlanta, GA 30332, USA

**Abstract.** We proved global existence and uniqueness of classical solutions to the initial boundary value problem for the 3D damped compressible Euler equations on bounded domain with slip boundary condition when the initial data is near its equilibrium. Time asymptotically, the density is conjectured to satisfy the porous medium equation and the momentum obeys to the classical Darcy's law. Based on energy estimate, we showed that the classical solution converges to steady state exponentially fast in time. We also proved that the same is true for the related initial boundary value problem of porous medium equation and thus justified the validity of Darcy's law in large time.

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\*Email address: panrh@math.gatech.edu

# 1 Introduction

We consider the 3D compressible Euler equations with frictional damping:

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0 \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) + \nabla P = -\alpha \rho U. \end{cases} \quad (1.1)$$

Such a system occurs in the mathematical modelling of compressible flow through a porous medium. Here  $\rho$ ,  $U$ ,  $M = \rho U$  and  $P$  denote the density, velocity, momentum and pressure respectively; the constant  $\alpha > 0$  models friction. Assuming the flow is a polytropic perfect gas, then  $P(\rho) = P_0 \rho^\gamma$ ,  $1 < \gamma$ , with  $P_0$  a positive constant, and  $\gamma$  the adiabatic gas exponent. Without loss of generality, we take  $P_0 = \frac{1}{\gamma}$ ,  $\alpha = 1$  throughout this paper. The system (1.1) is supplemented by the following initial and boundary conditions:

$$\begin{cases} (\rho, U)(\mathbf{x}, 0) = (\rho_0, U_0)(\mathbf{x}), \quad \mathbf{x} = (x, y, z) \in \Omega, \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \\ \int_{\Omega} \rho_0 d\mathbf{x} = \bar{\rho} > 0, \end{cases} \quad (1.2)$$

where  $\Omega \subset \mathbb{R}^3$  is a bounded domain with smooth boundary  $\partial\Omega$ ,  $\mathbf{n}$  is the unit outward normal vector on the boundary of  $\Omega$  and the last condition is imposed to avoid the trivial case,  $\rho \equiv 0$ .

The 1D version of (1.1) subject to various initial and initial-boundary conditions have been studied intensively during the past decades, both classical and weak solutions were constructed, and long time behaviors of different solutions were investigated. Extensive literatures are available for both Cauchy problem and initial-boundary value problem. In the direction of Cauchy problems, the readers are referred to [24], [11], [12] and [10] for existence of small smooth solutions; to [20], [6], and [7] for solutions in  $BV$ ; to [8] and [17] for  $L^\infty$  solutions. For large time behavior of solutions, we refer [11], [12], [25], [26] and [37] for small smooth solutions; and we refer [16], [18] and [38] for weak solutions. For initial boundary value problems, see [14], [21] and [27] for small smooth solutions; to [30] for  $L^\infty$  solutions. There are also some results on non-isentropic flows, see [13], [15], [22], [28] and [29].

From the physical point of view, the 3D model (1.1) describes more realistic phenomena. Also, the 3D compressible Euler equations carry some unique features, such as the effect of vorticity, which are totally absent in the 1D case and make the problem more mathematically challenging. Thus, due to strong physical background and significant mathematical challenge, system (1.1) and its time-asymptotic behavior are of great

importance and are much less understood than its 1D companion. For Cauchy problem of (1.1), investigations were carried out among small smooth solutions and we refer the readers to [32] and [34]. In the direction of initial-boundary value problems of (1.1), even the global existence of classical solutions is still open, especially when the boundary condition is non-characteristic, such as (1.2)<sub>2</sub>.

Due to the dissipation in the momentum equations and the boundary effect, the kinetic energy is expected to vanish as time tends to infinity while the potential energy will converge to a constant. Furthermore, it is easy to see that

$$\int_{\Omega} \rho(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} \rho_0(\mathbf{x}) d\mathbf{x} = \bar{\rho}$$

due to the conservation of total mass. This suggests that the asymptotic state of the solution should be  $(\rho, U)|_{t \rightarrow \infty} = (\bar{\rho}/|\Omega|, \mathbf{0})$ . In this paper, we will first prove, under the assumption that the initial perturbation around equilibrium state is small, there exists a unique global classical solution to (1.1)-(1.2) and the solution converges exponentially to equilibrium state.

For large time, it is conjectured that Darcy's law is valid and (1.1) is well approximated by the decoupled system

$$\begin{cases} \tilde{\rho}_t = \Delta P(\tilde{\rho}), \\ \tilde{M} = -\nabla P(\tilde{\rho}). \end{cases} \quad (1.3)$$

Where, the first equation is the well-known porous medium equation while the remaining equations state Darcy's law. The initial boundary conditions turn into

$$\begin{cases} \tilde{\rho}(\mathbf{x}, 0) = \tilde{\rho}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ (\nabla P(\tilde{\rho})) \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0. \end{cases} \quad (1.4)$$

We then prove that the solution of the related diffusion problem (1.3)-(1.4) tends to the same equilibrium state exponentially in time provided that

$$\int_{\Omega} \tilde{\rho}_0 d\mathbf{x} = \int_{\Omega} \rho_0 d\mathbf{x}. \quad (1.5)$$

We thus justified the validity of Darcy's law in large time.

Throughout this paper  $\|\cdot\|$  and  $\|\cdot\|_s$  denote the norms of  $L^2(\Omega)$  and  $H^s(\Omega)$  respectively, i.e.,

$$\begin{aligned} \|u\| &\equiv \|u\|_{L^2(\Omega)} = \left( \int_{\Omega} |u|^2 d\mathbf{x} \right)^{1/2}, \quad \text{for } u \in L^2(\Omega), \\ \|u\|_s &\equiv \|u\|_{H^s(\Omega)} = \left( \sum_{|\alpha| \leq s} \int_{\Omega} |D^\alpha u|^2 d\mathbf{x} \right)^{1/2}, \quad \text{for } u \in H^s(\Omega), \end{aligned}$$

where  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  is any multiindex with order  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$  and  $D^\alpha = \partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3}$ . For any vector valued function  $F = (f_1, f_2, f_3) : \Omega \rightarrow \mathbb{R}^3$ ,  $\|F\|_s^2 \equiv \|f_1\|_s^2 + \|f_2\|_s^2 + \|f_3\|_s^2$ . The energy space under consideration is:

$$X_3([0, T], \Omega) \equiv \{F : \Omega \times [0, T] \rightarrow \mathbb{R}^3(\text{or } \mathbb{R}) \mid \partial_t^l F \in L^\infty([0, T]; H^{3-l}(\Omega)), l = 0, 1, 2, 3\},$$

equipped with norm

$$\|F\|_{3,T} \equiv \text{ess sup}_{0 \leq t \leq T} \|F(\cdot, t)\| \equiv \text{ess sup}_{0 \leq t \leq T} \left[ \sum_{l=0}^3 \|\partial_t^l F(\cdot, t)\|_{3-l}^2 \right]^{1/2},$$

for any  $F \in X_3([0, T], \Omega)$ . Unless specified,  $C$  will denote a generic constant which is independent of time.

In this paper, we generalize the study of [14] and [15] on bounded domain with typical physical boundary condition (1.2)<sub>2</sub>. For the global existence and large time behavior of classical solutions, we have the following

**Theorem 1.** *Suppose that the initial data satisfy the compatibility condition, i.e.,  $\partial_t^l U(0) \cdot \mathbf{n}|_{\partial\Omega} = 0, 0 \leq l \leq 2$ , where  $\partial_t^l U(0)$  is the  $l^{\text{th}}$  time derivative at  $t = 0$  of any solution of (1.1)-(1.2), as calculated from (1.1) to yield an expression in terms of  $\rho_0$  and  $U_0$ . Then there exists a constant  $\varepsilon$  such that if  $(\rho_0 - \bar{\rho}/|\Omega|, U_0) \in H^3(\Omega)$  and  $\|(\rho_0 - \bar{\rho}/|\Omega|, U_0)\|_3 \leq \varepsilon$ , then there exists a unique global solution  $(\rho, U)$  of the initial-boundary value problem (1.1)-(1.2) in  $C^1(\bar{\Omega} \times [0, \infty)) \cap X_3([0, \infty), \Omega)$ . Moreover, there exist positive constants  $C > 0, \eta > 0$ , which are independent of  $t$ , such that*

$$\|(\rho - \bar{\rho}/|\Omega|)(\cdot, t)\| + \|U(\cdot, t)\| \leq C \|(\rho_0 - \bar{\rho}/|\Omega|, U_0)\|_3 \exp\{-\eta t\}. \quad (1.6)$$

Concerning the relationship between the solution of (1.1)-(1.2) and that of (1.3)-(1.5), we have

**Theorem 2.** *Let  $(\rho, U)$  be the unique global classical solution of (1.1)-(1.2) and define  $M = \rho U$ . Let  $(\tilde{\rho}, \tilde{M})$  be the global solution of (1.3)-(1.5) with  $\tilde{\rho}_0 \in L^\infty(\Omega)$ , and  $0 \leq \tilde{\rho}_0 \leq \rho^*$  for some constant  $\bar{\rho}/|\Omega| < \rho^* < \infty$ . Then, there exist constants  $C, \delta > 0$  independent of  $t$  such that*

$$\|(\rho - \tilde{\rho})(\cdot, t)\|_1 + \|(M - \tilde{M})(\cdot, t)\| \leq C \exp\{-\delta t\}, \quad \text{as } t \rightarrow \infty. \quad (1.7)$$

We prove Theorem 1 by showing the global existence and large time behavior of classical solutions to the IBVP for the perturbation  $(\rho - \bar{\rho}/|\Omega|, U - \mathbf{0})$ . Due to the slip boundary condition, the classical energy estimates can not be applied directly to spatial derivatives. The proof of Theorem 1 is based on some special energy estimates which strongly depend on the estimate of  $\nabla U$  by  $\nabla \times U$  and  $\nabla \cdot U$ , see Lemma 3.2 below. Using the special structure of (1.1) together with an induction on the number of spatial derivatives, the estimate of total energy is reduced to those for the vorticity and temporal derivatives. And the proof is completed by showing that (1.6) is true for the vorticity and temporal derivatives. Compared with the classical energy estimate for 3D initial-boundary value problems, which requires the localization of  $\partial\Omega$ , see for example [23, 31], our approach is short and neat. This idea has also been used for the incompressible Euler equations in a free boundary problem, see [5].

To prove Theorem 2, instead of comparing the two solutions directly, we take a detour. From (1.6) we see clearly that the classical solution of (1.1)-(1.2) tends to the constant state exponentially as time goes to infinity. We then prove that the same is true for the solution of (1.3)-(1.5). Thus, Theorem 2 is an easy consequence of the triangle inequality. Moreover, the proof of the asymptotic behavior of the solution of (1.3)-(1.5), see Theorem 4.1 below, requires neither smoothness nor smallness condition on the initial data, i.e., the initial perturbation around the asymptotic state could be rough and large, which is a significant difference from the proof of Theorem 1. The argument is somewhat delicate mainly due to the nonlinearity in the diffusion. It should be pointed out that, the global existence and large time behavior of solutions of (1.3)-(1.5) have been studied in [1] based on dynamical system approach, see also [33]. In this paper, we give a different proof on the asymptotic behavior of the solution based on the method of energy estimate. The decay in momentum is also achieved.

The plan of the rest of this paper is as follows. In Section 2, we reformulate the original system to get a quasi-linear symmetric hyperbolic system and give some basic facts that will be used in this paper together with the local existence result. In Section 3, we prove Theorem 1 by energy estimate. Finally, we investigate the large time behavior of the solution of (1.3)-(1.5) to show the close relationship between the two systems in Section 4.

## 2 Reformulation and local existence

In this section, we are going to reformulate the initial-boundary value problem (1.1)-(1.2). Without any loss of generality, we assume  $\bar{\rho}/|\Omega| = 1$ . First we reformulate (1.1) to get a symmetric hyperbolic system. Introducing the nonlinear transformation  $\tilde{\sigma} = \rho^\theta/\theta$  with  $\theta = (\gamma - 1)/2$  we get from the original system that

$$\begin{cases} \tilde{\sigma}_t + U \cdot \nabla \tilde{\sigma} + \theta \tilde{\sigma} \nabla \cdot U = 0 \\ U_t + U \cdot \nabla U + \theta \tilde{\sigma} \nabla \tilde{\sigma} = -U. \end{cases}$$

Since the equilibrium density is conjectured to be  $\bar{\rho}/|\Omega| = 1$ , we let  $\sigma = \tilde{\sigma} - 1/\theta$  and get the desired symmetric system for the perturbation

$$\begin{cases} \sigma_t + U \cdot \nabla \sigma + \theta \sigma \nabla \cdot U + \nabla \cdot U = 0 \\ U_t + U \cdot \nabla U + \theta \sigma \nabla \sigma + \nabla \sigma = -U. \end{cases} \quad (2.1)$$

The initial and boundary conditions become

$$\begin{cases} (\sigma, U)(\mathbf{x}, 0) = (\sigma_0, U_0)(\mathbf{x}), \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \quad (2.2)$$

with

$$\sigma_0 = \frac{\rho_0^\theta}{\theta} - \frac{1}{\theta}.$$

The following lemmas are consequences of regularity and could be proved using the same idea in [32].

**Lemma 2.1.** *For any  $T > 0$ , if  $(\rho, U) \in C^1(\bar{\Omega} \times [0, T])$  is a solution of (1.1) with  $\rho > 0$ , then  $(\sigma, U) \in C^1(\bar{\Omega} \times [0, T])$  is a solution of (2.1) with  $((\gamma - 1)/2)\sigma + 1 > 0$ . Conversely, if  $(\sigma, U) \in C^1(\bar{\Omega} \times [0, T])$  is a solution of (2.1) with  $((\gamma - 1)/2)\sigma + 1 > 0$  and  $\rho = (((\gamma - 1)/2)\sigma + 1)^{2/(\gamma-1)}$ , then  $(\rho, U) \in C^1(\bar{\Omega} \times [0, T])$  is a solution of (1.1) with  $\rho > 0$ .*

**Lemma 2.2.** *If  $(\rho, U) \in C^1(\bar{\Omega} \times [0, T])$  is a uniformly bounded solution of (1.1) with  $\rho(x, 0) > 0$ , then  $\rho(x, t) > 0$  on  $\bar{\Omega} \times [0, T]$ . If  $(\sigma, U) \in C^1(\bar{\Omega} \times [0, T])$  is a uniformly bounded solution of (2.1) with  $((\gamma - 1)/2)\sigma(x, 0) + 1 > 0$ , then  $((\gamma - 1)/2)\sigma(x, t) + 1 > 0$  on  $\bar{\Omega} \times [0, T]$ .*

The following local existence result can be established using the arguments in [31].

**Lemma 2.3. (Local existence)** *If  $(\sigma_0, U_0) \in H^3(\Omega)$  and satisfy the compatibility condition, i.e.,  $\partial_t^l U(0) \cdot \mathbf{n}|_{\partial\Omega} = 0, 0 \leq l \leq 2$ , then there exists a unique local solution*

$(\sigma, U)$  of the initial-boundary value problem (2.1)-(2.2) in  $C^1(\bar{\Omega} \times [0, T]) \cap X_3([0, T], \Omega)$  for some finite  $T > 0$ . Moreover, there exist positive constants  $\varepsilon_0, C_0(T)$  such that if  $\|\sigma(\cdot, 0)\|_3 + \|U(\cdot, 0)\|_3 \leq \varepsilon_0$ , then  $\|\sigma\|_{3,T} + \|U\|_{3,T} \leq C_0(\|\sigma(\cdot, 0)\|_3 + \|U(\cdot, 0)\|_3)$ .

### 3 Global existence and Large time behavior

In this section, we will prove the global existence and the large time behavior of the solution of (2.1)-(2.2). For convenience, we let

$$W(t) \equiv \|\sigma(t)\|_3^2 + \|U(t)\|_3^2 = \sum_{l=0}^3 (\|\partial_t^l \sigma(t)\|_{3-l}^2 + \|\partial_t^l U(t)\|_{3-l}^2). \quad (3.0)$$

**Theorem 3.1.** *There exists  $\varepsilon > 0$  such that if  $W(0) \leq \varepsilon^2$ , then there is a unique global classical solution of (2.1)-(2.2) such that there exist positive constants  $C > 0, \eta > 0$ , which are independent of  $t$ , such that*

$$W(t) \leq CW(0)e^{-\eta t}. \quad (3.1)$$

The proof of Theorem 3.1 is based on several steps of careful energy estimates which are stated as a sequence of lemmas. First we recall some inequalities of Sobolev type (c.f. [9]).

**Lemma 3.1.** *Let  $\Omega$  be any bounded domain in  $\mathbb{R}^3$  with smooth boundary. Then*

- (i)  $\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{H^2(\Omega)}$ ,
- (ii)  $\|f\|_{L^p(\Omega)} \leq C\|f\|_{H^1(\Omega)}, \quad 2 \leq p \leq 6$ ,

for some constant  $C > 0$  depending only on  $\Omega$ .

Due to the slip boundary condition, the spatial derivatives are unknown on the boundary. Following the standard procedure, see for example [23, 31], one can establish the energy estimates for the spatial derivatives by using cutoff functions and localizations of  $\partial\Omega$ , and Theorem 3.1 could be established in this fashion. However, we notice that the proof is long and tedious. In this paper we will give another version of the proof which is

short and neat. The proof will strongly depend on the following lemma (see [3]), which gives the estimate of  $\nabla U$  by  $\nabla \cdot U$  and  $\nabla \times U$ .

**Lemma 3.2.** *Let  $U \in H^s(\Omega)$  be a vector-valued function satisfying  $U \cdot \mathbf{n}|_{\partial\Omega} = 0$ , where  $\mathbf{n}$  is the unit outer normal of  $\partial\Omega$ . Then*

$$\|U\|_s \leq C(\|\nabla \times U\|_{s-1} + \|\nabla \cdot U\|_{s-1} + \|U\|_{s-1}), \quad (3.2)$$

for  $s \geq 1$ , and the constant  $C$  depends only on  $s$  and  $\Omega$ .

The next lemma is an application of Lemma 3.2, which plays an important role in the proof of Theorem 3.1. Indeed, the lemma states that the spatial derivatives are bounded by the temporal derivatives and the vorticity. Let  $\omega = \nabla \times U$  and define

$$E(t) \equiv \sum_{l=0}^3 (\|\partial_t^l \sigma\|^2 + \|\partial_t^l U\|^2), \quad \text{and} \quad V(t) \equiv \sum_{l=0}^2 \|\partial_t^l \omega\|_{2-l}^2, \quad (3.3)$$

**Lemma 3.3.** *Let  $(\sigma, U)$  be the solution of (2.1)-(2.2). There is a small constant  $\bar{\delta}$  such that if  $W(t) \leq \bar{\delta}$ , then there exists a constant  $C_1 > 0$  such that*

$$W(t) \leq C_1 (V(t) + E(t)). \quad (3.4)$$

**Proof.** From the velocity equation (2.1)<sub>2</sub> we have

$$\nabla \sigma = -\frac{1}{\theta\sigma + 1}(U + U_t + U \cdot \nabla U). \quad (3.5)$$

Taking the  $L^2$  inner product of (3.5) with  $\nabla \sigma$ , we get

$$\|\nabla \sigma\|^2 = \int_{\Omega} -\frac{1}{\theta\sigma + 1}(U + U_t + U \cdot \nabla U) \cdot \nabla \sigma \, d\mathbf{x},$$

using the smallness of  $W(t)$ , Lemma 3.1 (i), and Cauchy-Schwarz inequality, we easily get

$$\begin{aligned} \|\nabla \sigma\|^2 &\leq C(\|U\|^2 + \|U_t\|^2) + C\|U\|_{L^\infty}^2 \|\nabla U\|^2 \\ &\leq C(\|U\|^2 + \|U_t\|^2) + CW(t)^{\frac{3}{2}}. \end{aligned} \quad (3.6)$$

The continuity equation (2.1)<sub>1</sub> implies

$$\nabla \cdot U = -\frac{1}{\theta\sigma + 1}(\sigma_t + U \cdot \nabla \sigma). \quad (3.7)$$



Therefore, we obtain

$$\|\nabla \cdot U\|^2 \leq C \left( \|\sigma_t\|^2 + W(t)^{\frac{3}{2}} \right). \quad (3.8)$$

Using Lemma 3.2 with  $s = 1$  and (3.8) we have

$$\begin{aligned} \|U\|_1^2 &\leq C(\|\omega\|^2 + \|\nabla \cdot U\|^2 + \|U\|^2) \\ &\leq C(\|\omega\|^2 + \|\sigma_t\|^2 + \|U\|^2 + W(t)^{\frac{3}{2}}). \end{aligned} \quad (3.9)$$

Next, we take time derivatives of (3.5) and (3.7). It is clear that every time derivative up to order two of  $\nabla\sigma$  and  $\nabla \cdot U$  is again bounded by  $E(t)$ . Furthermore, together with an induction on the number of spatial derivatives, the same is true for any derivative up to order two of  $\nabla\sigma$  and  $\nabla \cdot U$ . By applying Lemma 3.2 with  $s = 1, 2, 3$  respectively we finally deduce the lemma. This completes the proof of Lemma 3.3.

Lemma 3.3 reduced the estimate of  $W(t)$  to those for  $E(t)$  and  $V(t)$ . Our next goal is to deal with the estimates of  $E(t)$  and  $V(t)$ .

**Lemma 3.4.** *There is a constant  $C > 0$  such that*

$$\frac{d}{dt}E(t) + 2 \sum_{l=0}^3 \|\partial_t^l U\|^2 \leq CW(t)^{\frac{3}{2}}. \quad (3.10)$$

**Proof.** *Zero order estimate:* We calculate  $\sigma(2.1)_1 + U \cdot (2.1)_2$  and get

$$\frac{1}{2} \frac{d}{dt}(\sigma^2 + |U|^2) + |U|^2 = -(1+\theta)\sigma(U \cdot \nabla\sigma) - \theta\sigma^2(\nabla \cdot U) - U \cdot (U \cdot \nabla U) - \nabla \cdot (\sigma U). \quad (3.11)$$

Integrating (3.11) over  $\Omega$  using the Divergence Theorem and the boundary condition we get

$$\frac{1}{2} \frac{d}{dt}(\|\sigma\|^2 + \|U\|^2) + \|U\|^2 \leq C(\|\nabla\sigma\|_{L^\infty} + \|\nabla U\|_{L^\infty})(\|\sigma\|^2 + \|U\|^2). \quad (3.12)$$

Applying Lemma 3.1 (i) to (3.12) we get

$$\frac{d}{dt}(\|\sigma\|^2 + \|U\|^2) + 2\|U\|^2 \leq CW(t)^{\frac{3}{2}}. \quad (3.13)$$

*First order estimate:* Differentiating (2.1) with respect to  $t$ , multiplying the resulting equations by  $\sigma_t, U_t$  respectively, we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt}(\sigma_t^2 + |U_t|^2) + |U_t|^2 \\ &= (1/2 - \theta)\sigma_t^2(\nabla \cdot U) - \sigma_t(U_t \cdot \nabla\sigma) - U_t \cdot (U_t \cdot \nabla U) - \frac{1}{2}|U_t|^2(\nabla \cdot U) \\ &\quad - \nabla \cdot \left( \frac{(\sigma_t^2 + |U_t|^2)}{2} U + (\theta\sigma + 1)\sigma_t U_t \right). \end{aligned}$$

Integrating the above equation over  $\Omega$  using the boundary conditions  $U \cdot \mathbf{n}|_{\partial\Omega} = 0$  and  $U_t \cdot \mathbf{n}|_{\partial\Omega} = 0$  we get

$$\frac{d}{dt} (\|\sigma_t\|^2 + \|U_t\|^2) + 2\|U_t\|^2 \leq C(\|\nabla\sigma\|_{L^\infty} + \|\nabla \cdot U\|_{L^\infty} + \|\nabla U\|_{L^\infty})W(t), \quad (3.14)$$

for some constant  $C > 0$ . From Lemma 3.1 (i) we get

$$\frac{d}{dt} (\|\sigma_t\|^2 + \|U_t\|^2) + 2\|U_t\|^2 \leq CW(t)^{\frac{3}{2}}. \quad (3.15)$$

*Second order estimate:* Repeating the above procedure again for 2nd order time derivatives we get the following

$$\frac{d}{dt} (\|\sigma_{tt}\|^2 + \|U_{tt}\|^2) + 2\|U_{tt}\|^2 \leq CW(t)^{\frac{3}{2}}; \quad (3.16)$$

*Third order estimate:* If we repeat the above procedure to the 3rd order estimates, we find that the 4th order estimates will be needed due to the Sobolev inequality in Lemma 3.1 (i). However, this issue could be resolved by Lemma 3.1 (ii). We calculate  $\partial_t^3 \sigma \partial_t^3 (2.1)_1 + \partial_t^3 U \cdot \partial_t^3 (2.1)_2$  and get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\sigma_{ttt}^2 + |U_{ttt}|^2) + |U_{ttt}|^2 = & \left[ \frac{1}{2} (\nabla \cdot U) (|U_{ttt}|^2) + \left(\frac{1}{2} - \theta\right) (\nabla \cdot U) \sigma_{ttt}^2 - \right. \\ & (U_{ttt} \cdot \nabla \sigma + 3U_t \cdot \nabla \sigma_{tt} + 3\theta \sigma_t \nabla \cdot U_{tt}) \sigma_{ttt} - \\ & \left. (U_{ttt} \cdot \nabla U + 3U_t \cdot \nabla U_{tt} + 3\theta \sigma_t \nabla \sigma_{tt}) \cdot U_{ttt} \right] - \\ & \left\{ 3(U_{tt} \cdot \nabla \sigma_t + \theta \sigma_{tt} \nabla \cdot U_t) \sigma_{ttt} + \right. \\ & \left. 3(U_{tt} \cdot \nabla U_t + \theta \sigma_{tt} \nabla \sigma_t) \cdot U_{ttt} \right\} - \\ & \nabla \cdot \left( \frac{\sigma_{ttt}^2}{2} U + \frac{|U_{ttt}|^2}{2} U + (\theta \sigma) \sigma_{ttt} U_{ttt} \right). \end{aligned}$$

Integrating the above equation over  $\Omega$ , applying Lemma 3.1 (i) to the terms inside the [ ] we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\sigma_{ttt}\|^2 + \|U_{ttt}\|^2) + \|U_{ttt}\|^2 \leq & CW(t)^{\frac{3}{2}} + 3 \left| \int_{\Omega} \sigma_{ttt} (U_{tt} \cdot \nabla \sigma_t + \theta \sigma_{tt} \cdot \nabla U_t) dx \right| \\ & + 3 \left| \int_{\Omega} U_{ttt} \cdot (U_{tt} \cdot \nabla U_t + \theta \sigma_{tt} \nabla \sigma_t) dx \right|. \end{aligned}$$

Using Hölder's inequality, Lemma 3.1 (ii), and Cauchy-Schwarz inequality we can estimate

the second term on the right hand side above as follows:

$$\begin{aligned}
& \left| \int_{\Omega} \sigma_{ttt} (U_{tt} \cdot \nabla \sigma_t + \theta \sigma_{tt} \cdot \nabla U_t) dx \right| \\
& \leq \|\sigma_{ttt}\|_{L^2} (\|U_{tt}\|_{L^4} \|D\sigma_t\|_{L^4} + \theta \|\sigma_{tt}\|_{L^4} \|DU_t\|_{L^4}) \\
& \leq C \|\sigma_{ttt}\|_{L^2} (\|U_{tt}\|_{H^1} \|D\sigma_t\|_{H^1} + \theta \|\sigma_{tt}\|_{H^1} \|DU_t\|_{H^1}) \\
& \leq C \|\sigma_{ttt}\|_{L^2} (\|U_{tt}\|_{H^1}^2 + \|D\sigma_t\|_{H^1}^2 + \|\sigma_{tt}\|_{H^1}^2 + \|DU_t\|_{H^1}^2) \\
& \leq CW(t)^{\frac{3}{2}}.
\end{aligned} \tag{3.17}$$

The third term can be estimated in the same way. Then we get the 3rd order estimate:

$$\frac{d}{dt} (\|\sigma_{ttt}\|^2 + \|U_{ttt}\|^2) + 2\|U_{ttt}\|^2 \leq CW(t)^{\frac{3}{2}}. \tag{3.18}$$

Therefore, (3.10) follows from (3.13), (3.15)-(3.16) and (3.18). This completes the proof of Lemma 3.4.

Lemma 3.4 contains the dissipation in velocity. In the next lemma we are going to explore the dissipation in density due to nonlinearity.

**Lemma 3.5.** *There exist constants  $c_0, C > 0$  such that*

$$\frac{d}{dt} \left( \sum_{l=1}^3 \int_{\Omega} (-\partial_t^{l-1} \sigma \partial_t^l \sigma) dx \right) + \sum_{l=0}^3 \|\partial_t^l \sigma\|^2 \leq CW(t)^{\frac{3}{2}} + c_0 \sum_{l=0}^3 \|\partial_t^l U\|^2. \tag{3.19}$$

**Proof.** First of all, due to the conservation of total mass we know  $\int_{\Omega} (\rho - 1) dx = 0$ , where  $\rho$  is the solution of (1.1) and  $1 = \bar{\rho}/|\Omega|$  is the equilibrium state of  $\rho$ . Letting  $\hat{\rho} = \rho - 1$ , then Poincaré's inequality (c.f. [9]) implies that  $\|\hat{\rho}\|^2 \leq C \|\nabla \hat{\rho}\|^2$ . By definition,  $\sigma = (\tau \hat{\rho} + 1) \hat{\rho}$  for some  $\tau \in [0, 1]$  and  $\nabla \sigma = (\hat{\rho} + 1)^{\theta-1} \nabla \hat{\rho}$ . So that for  $W(t)$  small,  $\|\sigma\|^2 \leq C \|\nabla \sigma\|^2$ . Using (3.6) we obtain

$$\|\sigma\|^2 \leq C(W(t)^{\frac{3}{2}} + \|U\|^2 + \|U_t\|^2). \tag{3.20}$$

Calculating  $\partial_t(2.1)_1 - (\theta\sigma + 1)\nabla \cdot (2.1)_2$  we get

$$\sigma_{tt} + (U \cdot \nabla \sigma)_t + \theta \sigma_t (\nabla \cdot U) - (\theta\sigma + 1) \nabla \cdot [U \cdot \nabla U + (\theta\sigma + 1) \nabla \sigma + U] = 0. \tag{3.21}$$

Multiplying (3.21) by  $\sigma$  we obtain

$$\begin{aligned}
& \sigma_{tt}\sigma + (U \cdot \nabla \sigma)_t \sigma + \theta \sigma_t (\nabla \cdot U) \sigma - (\theta \sigma + 1) \nabla \cdot [U \cdot \nabla U + (\theta \sigma + 1) \nabla \sigma + U] \sigma \\
&= (\sigma \sigma_t)_t - \sigma_t^2 + (U \cdot \nabla \sigma)_t \sigma + \theta \sigma \sigma_t (\nabla \cdot U) + (\theta \sigma + 1) \sigma \nabla \cdot U_t \\
&= (\sigma \sigma_t)_t - \sigma_t^2 + (U_t \cdot \nabla \sigma) \sigma + (U \cdot \nabla \sigma_t) \sigma + \theta \sigma \sigma_t (\nabla \cdot U) + \\
& \quad \nabla \cdot [(\theta \sigma^2 + \sigma) U_t] - U_t \cdot \nabla (\theta \sigma^2 + \sigma) \\
&= (\sigma \sigma_t)_t - \sigma_t^2 + (U_t \cdot \nabla \sigma) \sigma + \nabla \cdot (\sigma \sigma_t U) - \sigma \sigma_t (\nabla \cdot U) - \sigma_t (U \cdot \nabla \sigma) + \theta \sigma \sigma_t (\nabla \cdot U) + \\
& \quad \nabla \cdot [(\theta \sigma^2 + \sigma) U_t] - U_t \cdot \nabla (\theta \sigma^2 + \sigma) \\
&= (\sigma \sigma_t)_t - \sigma_t^2 + (U_t \cdot \nabla \sigma) \sigma + (\theta - 1) \sigma \sigma_t (\nabla \cdot U) - \sigma_t (U \cdot \nabla \sigma) - 2\theta \sigma U_t \cdot \nabla \sigma - \\
& \quad U_t \cdot \nabla \sigma + \nabla \cdot [(\theta \sigma^2 + \sigma) U_t + \sigma \sigma_t U] \\
&= 0,
\end{aligned} \tag{3.22}$$

where we used the equation  $U \cdot \nabla U + (\theta \sigma + 1) \nabla \sigma + U = -U_t$ . Integrating (3.22) over  $\Omega$  and using Cauchy-Schwarz inequality we get

$$-\frac{d}{dt} \left( \int_{\Omega} \sigma \sigma_t d\mathbf{x} \right) + \|\sigma_t\|^2 \leq C(W(t)^{\frac{3}{2}} + \|\nabla \sigma\|^2 + \|U\|^2), \tag{3.23}$$

which together with (3.6) gives

$$-\frac{d}{dt} \left( \int_{\Omega} \sigma \sigma_t d\mathbf{x} \right) + \|\sigma_t\|^2 \leq C(W(t)^{\frac{3}{2}} + \|U\|^2 + \|U_t\|^2). \tag{3.24}$$

Next, we take time derivatives of (3.20) and (3.3). Similar derivations show that

$$\begin{aligned}
& -\frac{d}{dt} \left( \int_{\Omega} \sigma_t \sigma_{tt} d\mathbf{x} \right) + \|\sigma_{tt}\|^2 \leq C(W(t)^{\frac{3}{2}} + \|U_t\|^2 + \|U_{tt}\|^2), \\
& -\frac{d}{dt} \left( \int_{\Omega} \sigma_{tt} \sigma_{ttt} d\mathbf{x} \right) + \|\sigma_{ttt}\|^2 \leq C(W(t)^{\frac{3}{2}} + \|U_{tt}\|^2 + \|U_{ttt}\|^2),
\end{aligned}$$

which together with (3.20) and (3.24) deduce (3.19). This completes the proof of Lemma 3.5.

Now, we are ready to combine Lemma 3.4 and 3.5 to characterize the total dissipation. For this purpose, we let  $C_2 \equiv \max\{2, c_0\}$ , and define

$$\begin{aligned}
E_1(t) &\equiv C_2 E(t) - \sum_{l=1}^3 \int_{\Omega} (\partial_t^{l-1} \sigma \partial_t^l \sigma) d\mathbf{x} \\
&= C_2 \sum_{l=0}^3 (\|\partial_t^l \sigma\|^2 + \|\partial_t^l U\|^2) - \sum_{l=1}^3 \int_{\Omega} (\partial_t^{l-1} \sigma \partial_t^l \sigma) d\mathbf{x}.
\end{aligned} \tag{3.25}$$

It is easy to see that  $E_1(t) \geq 0$  for any  $t \geq 0$ . Then we have

**Lemma 3.6.** *There exist constants  $C_3, C > 0$  such that*

$$\frac{d}{dt}E_1(t) + C_3E(t) \leq CW(t)^{\frac{3}{2}}. \quad (3.26)$$

**Proof.**  $C_2 \times (3.10) + (3.19)$  yields

$$\frac{d}{dt}E_1(t) + c_0 \sum_{l=0}^3 \|\partial_t^l U\|^2 + \sum_{l=0}^3 \|\partial_t^l \sigma\|^2 \leq CW(t)^{\frac{3}{2}}. \quad (3.27)$$

Let  $C_3 = \min\{c_0, 1\}$ , then (3.26) follows directly from (3.27).

The next lemma is contributed to the estimate of  $V(t)$  defined in Lemma 3.3.

**Lemma 3.7.** For  $V(t)$  defined in Lemma 3.3, there exists a constant  $C > 0$  such that

$$\frac{d}{dt}V(t) + 2V(t) \leq CW(t)^{\frac{3}{2}}. \quad (3.28)$$

**Proof.** Taking the curl of the velocity equation of (2.1) we get

$$\omega_t + \omega = -U \cdot \nabla \omega + \omega \cdot \nabla U - \omega(\nabla \cdot U).$$

Let  $\partial$  denote any mixed time and spatial derivative of order  $0 \leq |\partial| \leq 2$ , then by taking any mixed derivative of the above equation, we get

$$\partial \omega_t + \partial \omega = \partial \{-U \cdot \nabla \omega + \omega \cdot \nabla U - \omega(\nabla \cdot U)\}.$$

Multiplying the above equation by  $\partial \omega$  and integrating the resulting equation by using the boundary condition, together with the standard energy estimate used in deriving (3.17), we get

$$\frac{1}{2} \frac{d}{dt} \|\partial \omega(t)\|^2 + \|\partial \omega(t)\|^2 \leq CW(t)^{\frac{3}{2}}.$$

Finally, we deduce the lemma by summing up the above inequality for all  $0 \leq |\partial| \leq 2$ . This completes the proof of Lemma 3.7.

**Proof of Theorem 3.1.** From (3.3), (3.25), and the definition of  $C_2$  we can easily see that  $E(t)$  and  $E_1(t)$  are equivalent, i.e., there exist constants  $c_1, c_2 > 0$  such that

$$c_1 E_1(t) \leq E(t) \leq c_2 E_1(t). \quad (3.29)$$

Then, by (3.26) and (3.29) we have

$$\frac{d}{dt}E_1(t) + c_1C_3E_1(t) \leq CW(t)^{\frac{3}{2}}. \quad (3.30)$$

Combining (3.28) and (3.30) we get

$$\frac{d}{dt}\left(V(t) + E_1(t)\right) + \left(2V(t) + c_1C_3E_1(t)\right) \leq CW(t)^{\frac{3}{2}}. \quad (3.31)$$

Let  $C_4 \equiv \min\{2, c_1C_3\}$ , then we get from (3.31) that

$$\frac{d}{dt}\left(V(t) + E_1(t)\right) + C_4\left(V(t) + E_1(t)\right) \leq CW(t)^{\frac{3}{2}}. \quad (3.32)$$

On the other hand, from (3.4) and (3.29) we see that

$$W(t) \leq C_1\left(V(t) + c_2E_1(t)\right). \quad (3.33)$$

Let  $C_5 \equiv \max\{C_1, c_2C_1\}$ , then we get

$$W(t) \leq C_5\left(V(t) + E_1(t)\right). \quad (3.34)$$

For  $W(t)$  sufficiently small, (3.32) and (3.34) yield

$$\frac{d}{dt}\left(V(t) + E_1(t)\right) + C_4\left(V(t) + E_1(t)\right) \leq \frac{C_4}{2}\left(V(t) + E_1(t)\right). \quad (3.35)$$

Thus, we get

$$\frac{d}{dt}\left(V(t) + E_1(t)\right) + \frac{C_4}{2}\left(V(t) + E_1(t)\right) \leq 0, \quad (3.36)$$

which yields the exponential decaying of  $V(t) + E_1(t)$ . Finally, the exponential decay of  $W(t)$  follows from (3.34). This completes the proof of Theorem 3.1.

## 4 Asymptotic behavior and Porous Medium Equation.

In this section, we investigate the large time behavior of classical solutions of (1.3)-(1.4). Suggested by Theorem 3.1, we see that the solution of (1.1)-(1.2) approaches the constant state  $(\bar{\rho}/|\Omega|, \mathbf{0})$  exponentially as time approaches infinity. As indicated in the

introduction, we also expect that (1.1)-(1.2) is captured by (1.3)-(1.4) time asymptotically if

$$\int_{\Omega} \tilde{\rho}_0 d\mathbf{x} = \int_{\Omega} \rho_0 d\mathbf{x} = \bar{\rho}.$$

In view of Theorem 3.1, we will show that the large time asymptotic state of (1.3)-(1.4) is also the constant state  $(\bar{\rho}/|\Omega|, \mathbf{0})$ . Then, by applying the triangle inequality we can prove Theorem 2. Without loss of generality, we assume  $\bar{\rho}/|\Omega| = 1$ .

Consider

$$\begin{cases} \tilde{\rho}_t = \Delta P(\tilde{\rho}), \\ \tilde{\rho}(\mathbf{x}, 0) = \tilde{\rho}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ (\nabla P(\tilde{\rho})) \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \quad (4.1)$$

where the initial data satisfy

$$\begin{cases} \int_{\Omega} \tilde{\rho}_0 d\mathbf{x} = \bar{\rho}, \quad \tilde{\rho}_0(\mathbf{x}) \in L^\infty(\Omega), \\ 0 \leq \tilde{\rho}_0(\mathbf{x}) \leq \rho^* \quad \text{for some constant } 1 < \rho^* < \infty. \end{cases} \quad (4.2)$$

The global existence of solutions to (4.1)-(4.2) has been established in [1], see also [33]. It is also shown in there that  $\|(\tilde{\rho} - 1)\|_{L^\infty}$  tends to zero exponentially as time goes to infinity. Here, we give a different proof based on the method of energy estimate including the decay in momentum.

**Theorem 4.1.** *Let  $\tilde{\rho}$  be the global solution of (4.1)-(4.2) with  $\tilde{M} = -\nabla P(\tilde{\rho})$ . Then, there exist positive constants  $C > 0, \eta > 0$  independent of  $t$  such that*

$$\|(\tilde{\rho} - 1)\|_1 + \|\tilde{M}(\cdot, t)\| \leq Ce^{-\eta t}, \quad \text{as } t \rightarrow \infty.$$

**Proof.** First, we observe that due to the comparison principle (c.f. [33]),

$$0 \leq \tilde{\rho}(\mathbf{x}, t) \leq \rho^*, \quad \forall (\mathbf{x}, t) \in \bar{\Omega} \times [0, \infty). \quad (4.3)$$

Second, there is a  $T > 0$  such that  $\tilde{\rho}(\mathbf{x}, t)$  is a classical solution and  $\tilde{\rho}(\mathbf{x}, t) > \frac{1}{2}$  for  $t > T$  and  $\mathbf{x} \in \bar{\Omega}$ , see [33]. Then, for  $t > T$ , we consider the equation

$$(\tilde{\rho} - 1)_t = \Delta(P(\tilde{\rho}) - P(1)). \quad (4.4)$$

Taking  $L^2$  inner product of (4.4) with  $(\tilde{\rho} - 1)$  we obtain, after integration by parts

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\tilde{\rho} - 1)\|^2 - \int_{\Omega} \Delta[P(\tilde{\rho}) - P(1)](\tilde{\rho} - 1) d\mathbf{x} \\ &= \frac{1}{2} \frac{d}{dt} \|(\tilde{\rho} - 1)\|^2 + \int_{\Omega} \frac{|\nabla(P(\tilde{\rho}) - P(1))|^2}{P'(\tilde{\rho})} d\mathbf{x} \\ &= 0. \end{aligned} \quad (4.5)$$

Using (4.3) we get from (4.5) that

$$\frac{1}{2} \frac{d}{dt} \|(\tilde{\rho} - 1)\|^2 + \frac{1}{P'(\rho^*)} \|\nabla(P(\tilde{\rho}) - P(1))\|^2 \leq 0. \quad (4.6)$$

Since  $\tilde{\rho} = \gamma^{1/\gamma} \tilde{P}^{1/\gamma}$ , for smooth solutions, (4.1)<sub>1</sub> is equivalent to

$$\tilde{P}_t - \gamma^{1-1/\gamma} \tilde{P}^{1-1/\gamma} \Delta \tilde{P} = 0, \quad (4.7)$$

where  $\tilde{P} = P(\tilde{\rho})$ . Now, we define

$$\Phi \equiv \tilde{P} - \bar{P} = P(\tilde{\rho}) - P(1),$$

then we get

$$\Phi_t - a \tilde{P}^{1-1/\gamma} (\Delta \Phi) = 0, \quad (4.8)$$

where  $a = \gamma^{1-1/\gamma}$ . Taking  $L^2$  inner product of (4.8) with  $\Delta \Phi$  we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla \Phi\|^2 + a P(1/2)^{1-1/\gamma} \|\Delta \Phi\|^2 \leq 0. \quad (4.9)$$

Combining (4.6) and (4.9) we deduce

$$\frac{1}{2} \frac{d}{dt} \left( \|(\tilde{\rho} - 1)\|^2 + \|\nabla \Phi\|^2 \right) + C_1 \left( \|\nabla \Phi\|^2 + \|\Delta \Phi\|^2 \right) \leq 0, \quad (4.10)$$

for  $C_1 = \min\{1/P'(\rho^*), aP(1/2)^{1-1/\gamma}\}$ .

To explore the secret of (4.10), we observe that since

$$\Phi = P(\tilde{\rho}) - P(1) = P'(\varrho)(\tilde{\rho} - 1)$$

for some  $\varrho \in [1/2, \rho^*]$ , then

$$\begin{aligned} \|\Phi\|^2 &\leq P'(\rho^*)^2 \|(\tilde{\rho} - 1)\|^2, \\ \|\nabla \Phi\|^2 &\geq P'(1/2)^2 \|\nabla(\tilde{\rho} - 1)\|^2. \end{aligned} \quad (4.11)$$



Due to the conservation of total mass, i.e.  $\int_{\Omega} (\tilde{\rho} - 1) d\mathbf{x} = 0$ , and Poincaré's inequality we get

$$\begin{aligned} \|\Phi\|^2 &\leq P'(\rho^*)^2 \|\tilde{\rho} - 1\|^2 \\ &\leq CP'(\rho^*)^2 \|\nabla(\tilde{\rho} - 1)\|^2 \\ &\leq C \left( \frac{P'(\rho^*)}{P'(1/2)} \right)^2 \|\nabla\Phi\|^2. \end{aligned} \tag{4.12}$$

Combining (4.10)–(4.12) we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|\tilde{\rho} - 1\|^2 + \|\nabla\Phi\|^2 \right) + C_2 \left( \|\tilde{\rho} - 1\|^2 + \|\nabla\Phi\|^2 + \|\Delta\Phi\|^2 \right) \leq 0, \tag{4.13}$$

for some constant  $C_2 > 0$  depending on  $\rho^*$ . Finally, we deduce the theorem by (4.13) and noticing that  $\tilde{M} = -\nabla\Phi$ . This completes the proof of Theorem 4.1.

Theorem 2 in Section 1 is a immediate consequence of Theorem 3.1 and Theorem 4.1.

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