GLOBAL BV SOLUTIONS FOR THE P-SYSTEM WITH FRICTIONAL DAMPING

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ABSTRACT. We construct global BV solutions to the Cauchy problem for the damped p-system, under initial data with distinct end-states. The solution will be realized as a perturbation of its asymptotic profile, in which the specific volume satisfies the porous media equation and the velocity obeys the classical Darcy law for gas flow through a porous medium.

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1 Introduction

We consider the damped p-system

$$\begin{cases} u_t - v_x = 0 \\ v_t + p(u)_x + v = 0, \end{cases} \tag{1.1}$$

which we regard as governing isentropic gas flow through a porous medium. In that connection, v stands for velocity, u denotes specific volume, p is pressure and $-v$ is the frictional force exerted on the gas by the porous medium. The (given) smooth function $p(u)$ is defined on $(0, \infty)$ and is strictly decreasing, $p'(u) < 0$. For example, in the case of a polytropic gas, we have $p(u) = u^{-\gamma}$, with $\gamma \geq 1$.

We study the Cauchy problem for (1.1) with prescribed initial data

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \qquad -\infty < x < \infty. \tag{1.2}$$

Even though the frictional force has a damping effect, the system is not strictly dissipative. This renders the question of global existence and large time behavior of solutions mathematically challenging and physically interesting. It is known that friction prevents the breaking of waves of small amplitude. Consequently, global classical solutions exist when u_0 and v_0 are C^1 and their derivatives u_0' and v_0' are sufficiently small [25]. However, when the maxima of $|u_0'|$ and/or $|v_0'|$ exceed certain threshold values, waves eventually break and shocks develop. The existence of global weak solutions in L^p can be established by the method of compensated compactness [5, 12, 13, 21, 28]. The aim here is to develop the theory in the BV setting.

Because of the nature of frictional damping, it is natural to conjecture that as $t \to \infty$ the inertial term v_t decays to zero faster than the other terms, in which case (u, v) shall be represented asymptotically by functions $(\tilde{u}, \tilde{v})$, where $\tilde{v}$ obeys the classical Darcy law

$$\tilde{v} = -p(\tilde{u})_x, \tag{1.3}$$

while $\tilde{u}$ satisfies the porous media equation

$$\tilde{u}_t + p(\tilde{u})_{xx} = 0. \tag{1.4}$$

In particular, (1.4) admits a family of self-similar solutions

$$\tilde{u}(x,t) = \Psi\left(\frac{x-\overline{x}}{\sqrt{t+1}}\right), \tag{1.5}$$

where Ψ satisfies the ordinary differential equation

$$\ddot{p}(\Psi(\xi)) - \tfrac{1}{2}\xi\dot{\Psi}(\xi) = 0, \tag{1.6}$$

and $\overline{x}$ is an arbitrary parameter.

Solutions of (1.6) on $(-\infty,\infty)$ are uniquely identified by their (positive) end-states $\Psi(\pm\infty) = u_\pm$. They are monotone functions that get rapidly flat as $\xi \to \pm\infty$: For $u_\pm$ confined in any fixed compact subinterval J of $(0,\infty)$,

$$\left|\frac{d^k\Psi}{d\xi^k}\right| \leq c\,|u_+ - u_-|\,e^{-\mu\xi^2}, \quad -\infty < \xi < \infty, \;\; k = 1, 2, 3, \tag{1.7}$$

where the positive constants c and μ depend solely on the maximum of p' and on the maxima of $|p'|$, $|p''|$ and $|p'''|$ over J. The conjecture is that any solution of (1.1), (1.2), with initial data (u_0, v_0) that are sufficiently flat at $x = \pm\infty$, will be represented asymptotically by one of the self-similar solutions $(\tilde{u}, \tilde{v})$ induced by Ψ through (1.5) and (1.3). This has indeed been established in the setting of smooth solutions [16, 17, 22, 26, 27, 30], as well as in the setting of admissible bounded weak solutions [18, 19, 20, 29]. The aim here is to address this question in the context of BV solutions.

Faced with the task of constructing BV solutions to systems of balance laws, one first attempts to employ some method that works well for systems of conservation laws, such as the Glimm scheme [15], the front tracking algorithm [4], or the vanishing viscosity approach [3], in conjunction with operator splitting to account for the effect of the source term. This procedure is indeed effective for strictly dissipative systems of balance laws [6, 11], in which the source acts as a damper in any wave interaction. However, in the case of our system (1.1), which is merely weakly dissipative [10], the source is acting as an amplifier in certain wave interactions, unless $p(u)$ belongs to a very restrictive class [1], which does not include the interesting case of polytropic gases, $p(u) = u^{-\gamma}, \gamma \geq 1$. Apart from these exceptional $p(u)$, construction of BV solutions to (1.1), (1.2) based solely on local wave interaction estimates has been achieved [23] only for $p(u) = u^{-1}$, by exploiting the fact that in this case it is possible to handle interactions of waves with large amplitude [24]. The observation that traveling waves of (1.1) propagate with "hyperbolic" characteristic speed whereas traveling waves of (1.4) move with much higher "parabolic" speed provides another indication of the difficulty to obtain uniform BV bounds by employing exclusively local estimates. Accordingly, one has to supplement local wave interaction estimates with global estimates supplied by entropy inequalities. This idea was implemented successfully [8] for constructing global BV solutions to (1.1), (1.2) in the special case where u_0 and v_0 have finite total mass. In that situation, the long time behavior is trivial, as solutions decay to zero. Our goal here is to employ a similar approach for treating the more interesting, albeit more challenging, case of initial data (u_0, v_0) that are allowed to approach distinct limits at $\pm\infty$:

$$(u_0(x), v_0(x)) \to (u_\pm, v_\pm), \text{ as } x \to \pm\infty. \tag{1.8}$$

The aim is to construct global BV solutions (u, v) to (1.1), (1.2), as perturbations $u = \tilde{u} + w, v = \tilde{v} + z$ of $\tilde{u}$ in the form (1.5) and $\tilde{v}$ induced by (1.3). The function Ψ will be the solution of (1.6) with end-states $\Psi(\pm\infty) = u_\pm = u_0(\pm\infty)$. The value of the parameter $\overline{x}$ will be dictated by the balance law for the total "mass" $\int u\, dx$. Existence of (w, z) in BV will be established under the assumption that $|u_+ - u_-|, |v_+ - v_-|$, as well as the L^1 norm, the L^2 norm and the total variation of the initial perturbation (w_0, z_0) are sufficiently small. It will be demonstrated that as $t \to \infty$, the perturbation (w, z) decays to zero in $L^r, 1 < r \leq 2$, at the rate $O(t^{-\frac{r-1}{r}})$.

2 The Main Result

In this section we lay the groundwork for stating the main theorem, pertaining to the existence and long time behavior of solutions to the Cauchy problem (1.1), (1.2), with initial data (u_0, v_0) that are functions of bounded variation on $(-\infty, \infty)$, attaining limits (1.8). For clarity, we shall treat the special case where $u_- \neq u_+$ but $v_- = v_+ = 0$. At the end of the section, the reader will find an outline of the minor modifications that would be needed for dealing with the general case where both $u_- \neq u_+$ and $v_- \neq v_+$.

We consider the solution Ψ of (1.6), with the same end-states $\Psi(\pm\infty) = u_\pm$ as u_0, which induces, through (1.5), the family $\tilde{u}(x,t)$ of self-similar solutions of (1.4). The corresponding $\tilde{v}(x,t)$ is then obtained from (1.3). We set $\tilde{u}_0(x) = \tilde{u}(x,0)$, $\tilde{v}_0(x) = \tilde{v}(x,0)$, and assume that u_0 and v_0 are sufficiently flat at $\pm\infty$ to render both $w_0 = u_0 - \tilde{u}_0$ and $z_0 = v_0 - \tilde{v}_0$ in $L^1(-\infty,\infty)$.

The parameter $\overline{x}$ in (1.5) will be fixed so that

$$\int_{-\infty}^{\infty} w_0(x)dx = \int_{-\infty}^{\infty} [u_0(x) - \tilde{u}_0(x)]\, dx = 0. \tag{2.1}$$

Consequently, the function ϕ_0, defined by

$$\phi_0(x) = \int_{-\infty}^{x} w_0(y)dy, \tag{2.2}$$

vanishes at $\pm\infty$. We assume w_0 is sufficiently flat at $\pm\infty$ to render ϕ_0 in $L^2(-\infty,\infty)$.

The size of the initial data will be measured by means of the parameters

$$\Lambda_0 = |u_+ - u_-|\,, \tag{2.3}$$

$$\Lambda_1 = \int_{-\infty}^{\infty} \{|w_0(x)| + |z_0(x)|\}dx, \tag{2.4}$$

$$\Lambda_2 = \int_{-\infty}^{\infty} \{w_0^2(x) + z_0^2(x) + \phi_0^2(x)\}dx, \tag{2.5}$$

$$\Lambda = \Lambda_0 + \Lambda_0^2 + \Lambda_0^3 + \Lambda_1 + \Lambda_2 + \Lambda_0\Lambda_2, \tag{2.6}$$

$$M = TVw_0(\cdot) + TVz_0(\cdot). \tag{2.7}$$

The solution (u, v) of (1.1), (1.2) will be estimated and constructed as a perturbation of $(\tilde{u}, \tilde{v})$:

$$\begin{cases} u(x,t) = \tilde{u}(x,t) + w(x,t) \\ v(x,t) = \tilde{v}(x,t) + z(x,t). \end{cases} \tag{2.8}$$

The component u of the solution will take values in an arbitrary, but fixed, compact subinterval J of $(0, \infty)$, which contains in its interior the points $u_\pm$.

By virtue of (1.4) and (1.3), (w, z) must satisfy the inhomogeneous system of balance laws

$$\begin{cases} w_t - z_x = 0 \\ z_t + [p(\tilde{u} + w) - p(\tilde{u})]_x + z = p(\tilde{u})_{xt}, \end{cases} \tag{2.9}$$

with initial conditions

$$w(x,0) = w_0(x), \quad z(x,0) = z_0(x), \quad -\infty < x < \infty. \tag{2.10}$$

Notice that w vanishes at $x = \pm\infty$, for all $t > 0$, and has zero total mass. Indeed,

$$\frac{d}{dt}\int_{-\infty}^{\infty} w(x,t)dx = z(\infty, t) - z(-\infty, t) = 0, \tag{2.11}$$

which implies, on account of (2.1),

$$\int_{-\infty}^{\infty} w(x,t)dx = 0, \qquad t \geq 0. \tag{2.12}$$

Consequently, the function ϕ defined by

$$\phi(x,t) = \int_{-\infty}^{x} w(y,t)dy \tag{2.13}$$

vanishes at $x = \pm\infty$, for all t. Notice that

$$\phi_x = w, \qquad \phi_t = z. \tag{2.14}$$

The following proposition describes the large time behavior of (w, z) in L^2:

Theorem 2.1. *Let (w, z) be an admissible weak solution of the Cauchy problem (2.9), (2.10) defined on a strip $(-\infty, \infty) \times [0, T)$, with $0 < T \leq \infty$, and such that $u = \tilde{u} + w$ takes values in the compact interval J. Then*

$$\int_{-\infty}^{\infty} \{w^2(x,t) + z^2(x,t) + \phi^2(x,t)\} dx + \int_0^t \int_{-\infty}^{\infty} \{w^2(x,\tau) + z^2(x,\tau)\} dx d\tau \leq K(\Lambda_0^2 + \Lambda_2), \tag{2.15}$$

$$(t+1) \int_{-\infty}^{\infty} \{w^2(x,t) + z^2(x,t)\} dx + \int_0^t \int_{-\infty}^{\infty} (\tau + 1) z^2(x,\tau) dx d\tau \leq K(\Lambda_0^2 + \Lambda_0^3 + \Lambda_2 + \Lambda_0 \Lambda_2), \tag{2.16}$$

for all $t \in [0, T)$, where K depends solely on the maximum and minimum values of $p'(u)$, and the maximum of $|p''(u)|$ on J.

Estimates akin to (2.15) and/or (2.16) were derived earlier in [16, 17, 26, 27], for smooth solutions with "small" initial data, and in [29], for bounded weak solutions obtained by the vanishing viscosity method. Both (2.15) and (2.16) will be established here, in Section 3, for any weak solution that conserves or dissipates (mechanical) energy.

Next we state a result on L^1 stability, which will be proved in Section 4:

Theorem 2.2. *There is $\rho > 0$ such that any admissible weak solution (w, z) of the Cauchy problem (2.9), (2.10) defined on a strip $(-\infty, \infty) \times [0, T)$, with $0 < T \leq \infty$, and taking values in the disk of radius ρ centered at the origin, satisfies*

$$\int_{-\infty}^{\infty} \{|w(x,t)| + |z(x,t)|\} dx \leq K\Lambda, \tag{2.17}$$

where K depends solely on the maximum and minimum values of $p'(u)$, and the maxima of $|p''(u)|$ and $|p'''(u)|$ on the interval $[\min u_\pm - \rho, \max u_\pm + \rho]$.

The decay rate, as $t \to \infty$, in L^r, for $1 < r \leq 2$, is obtained by using (2.16), (2.17) and Hölder's inequality:

Corollary 2.1. *Under the assumptions of Theorem* 2.2 *and for any* $r \in [1, 2]$,

$$\int_{-\infty}^{\infty} \{|w(x,t)|^r + |z(x,t)|^r\} dx \leq \frac{K\Lambda}{(t+1)^{r-1}} \,, \tag{2.18}$$

for all $t \in [0, T)$.

With the help of the estimate (2.17), we will construct, in Section 5, a global admissible BV solution (w, z) to the Cauchy problem (2.9), (2.10), which will induce, through (2.8), a global admissible BV solution (u, v) to the original Cauchy problem (1.1), (1.2):

Theorem 2.3. *There are positive constants* $\tilde{\Lambda}, \tilde{M}$ *and* ν *such that if* $\Lambda \leq \tilde{\Lambda}$ *and* $M \leq \tilde{M}$, *then there exists an admissible* BV *solution* (w, z) *of the Cauchy problem* (2.9), (2.10), *defined on the upper half-plane* $(-\infty, \infty) \times [0, \infty)$. *In addition to* (2.15), (2.16) *and* (2.17), (w, z) *satisfies the estimate*

$$TVw(\cdot, t) + TVz(\cdot, t) \leq KMe^{-\nu t} + K\Lambda, \tag{2.19}$$

for any $t > 0$, where K *depends solely on the maximum and minimum values of* $p'(u)$, and the maxima of $|p''(u)|$ *and* $|p'''(u)|$ *on a compact interval containing* u_- *and* u_+ *in its interior. Consequently,* (u, v) *defined through* (2.8) *is an admissible BV solution of the Cauchy problem* (1.1), (1.2) *on* $(-\infty, \infty) \times [0, \infty)$.

We close this section with an outline of the modifications that would be needed in order to deal with the general situation where both $u_- \neq u_+$ and $v_- \neq v_+$. The aim is to modify (2.8) in such a way that w and z still vanish at $x = \pm\infty$, and w still satisfies the normalization condition (2.12), in which case one may still define the potential ϕ through (2.13). To that end, the parameter $\overline{x}$ in (1.5) should now be selected so that, in the place of (2.1),

$$\int_{-\infty}^{\infty} [u_0(x) - \tilde{u}_0(x)] dx = -(v_+ - v_-). \tag{2.20}$$

Moreover, (2.8) should be replaced by

$$\begin{cases} u(x,t) = \tilde{u}(x,t) - (v_+ - v_-)m'(x)e^{-t} + w(x,t) \\ \\ v(x,t) = \tilde{v}(x,t) + [v_- + (v_+ - v_-)m(x)]e^{-t} + z(x,t), \end{cases} \tag{2.21}$$

where m is an arbitrary (but fixed) smooth function on $(-\infty, \infty)$, such that $m(x) = 0$ for $x < -1$ and $m(x) = 1$ for $x > 1$. To see the purpose of these changes, notice first that (2.20) together with $(2.21)_1$ still imply (2.12), because now $v(\pm\infty, t) = v_\pm e^{-t}$. Furthermore, $(2.21)_2$ yields $z(\pm\infty, t) = 0$. It turns out that under the above modifications the assertions of Theorems 2.1, 2.2 and 2.3 are still valid and the proofs remain essentially the same. Thus, for simplicity, in the sequel we shall deal exclusively with the special case $v_- = v_+ = 0$.

3 Stability in L^2

The aim here is to derive L^2 bounds on admissible weak solutions (w, z) of (2.9), (2.10) that will establish the estimates (2.15) and (2.16). Throughout this section, c will stand for a generic positive constant that depends solely on the maximum of $p'(u)$ and the maxima of $|p'(u)|$, $|p''(u)|$ and $|p'''(u)|$ on the compact interval J.

The effect of $\tilde{u}$ on solutions of (2.9) will be monitored with the help of the following bounds, which are easily derived from (1.5), (1.6) and (2.3):

$$|\tilde{u}_x(x,t)| \le c\Lambda_0(t+1)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\mu\xi^2), \tag{3.1}$$

$$|\tilde{u}_t(x,t)| \le c\Lambda_0(t+1)^{-1} \exp(-\tfrac{1}{2}\mu\xi^2), \tag{3.2}$$

$$|\partial_x^j \partial_t^i p(\tilde{u}(x,t))| \le c\Lambda_0(t+1)^{-(i+\frac{1}{2}j)} \exp(-\tfrac{1}{2}\mu\xi^2), \;\; 1 \le i+j \le 3, \tag{3.3}$$

where $\xi = (x - \overline{x})/\sqrt{t+1}$.

The requisite estimates will be obtained by means of entropy inequalities. A function $\eta(w,z;\tilde u)$ will be an entropy for the system (2.9), with associated entropy flux $q(w,z;\tilde u)$, if

$$\begin{cases} q_w(w,z;\tilde u) = p'(\tilde u + w)\eta_z(w,z;\tilde u) \\ \\ q_z(w,z;\tilde u) = -\eta_w(w,z;\tilde u). \end{cases} \tag{3.4}$$

Whenever the entropy $\eta(w,z;\tilde u)$ is a convex function of (w,z), admissible solutions of (2.9) satisfy the inequality

$$\eta_t + q_x + z\eta_z \le \eta_{\tilde u}\tilde u_t + \{q_{\tilde u} - [p'(\tilde u + w) - p'(\tilde u)]\eta_z\}\tilde u_x + \eta_z p(\tilde u)_{xt}\ , \tag{3.5}$$

in the sense of distributions.

Proof of Theorem 2.1. We employ the entropy-entropy flux pair[3]

$$\begin{cases} \tilde\eta(w,z;\tilde u) = z^2 - 2\int_0^w [p(\tilde u + \omega) - p(\tilde u)]d\omega \\ \\ \tilde q(w,z;\tilde u) = 2[p(\tilde u + w) - p(\tilde u)]z. \end{cases} \tag{3.6}$$

On the range of our solution, the Hessian of $\tilde\eta$ is uniformly positive definite and

$$z^2 + \alpha w^2 \le \tilde\eta(w,z;\tilde u) \le z^2 + \alpha^{-1}w^2, \tag{3.7}$$

for some $\alpha \in (0,1)$. The inequality (3.5) now takes the form

$$\tilde\eta_t + \tilde q_x + 2z^2 \le -2[p(\tilde u + w) - p(\tilde u) - p'(\tilde u)w]\tilde u_t + 2zp(\tilde u)_{xt}\ . \tag{3.8}$$

The above inequality shall be combined with an equation of quadratic order derived by multiplying $(2.9)_2$ by the Lipschitz function ϕ. Recalling (2.14), we obtain

$$\begin{aligned} &[\tfrac12\phi^2 + \phi z]_t + [\phi[p(\tilde u + w) - p(\tilde u) - p(\tilde u)_t]]_x - [p(\tilde u + w) - p(\tilde u)]w \\ &= z^2 - wp(\tilde u)_t\ . \end{aligned} \tag{3.9}$$

[3] Written in terms of the original state variables (u,v), $\frac12\tilde\eta$ is what is commonly called [10] a *relative entropy* of the system (1.1), associated with the energy $\frac12 v^2 + \int pdu$. Thus, the inequality (3.8), below, expresses mechanical energy dissipation.

Since $p(u)$ is strictly decreasing,

$$-[p(\tilde{u}+w)-p(\tilde{u})]w \geq 3\delta w^2 \tag{3.10}$$

with $\delta > 0$. We add (3.8) and (3.9). In the resulting inequality, we apply (3.10) as well as the estimates

$$-2[p(\tilde{u}+w)-p(\tilde{u})-p'(\tilde{u})w]\tilde{u}_t \leq \delta w^2 + c|\tilde{u}_t|^2, \tag{3.11}$$

$$-wp(\tilde{u})_t \leq \delta w^2 + c|p(\tilde{u})_t|^2, \tag{3.12}$$

$$2zp(\tilde{u})_{xt} \leq \tfrac{1}{2}z^2 + 2|p(\tilde{u})_{xt}|^2\ . \tag{3.13}$$

Thus, recalling (3.2) and (3.3), we conclude

$$\begin{aligned}[\tilde{\eta} + \tfrac{1}{2}\phi^2 + \phi z]_t + [\tilde{q} + \phi[p(\tilde{u}+w) - p(\tilde{u}) - p(\tilde{u})_t]]_x + \delta w^2 + \tfrac{1}{2}z^2 \\ \leq c\Lambda_0^2(t+1)^{-2}e^{-\mu\xi^2}.\end{aligned} \tag{3.14}$$

Integrating (3.14) over the strip $(-\infty,\infty)\times[0,t]$, for $t\in(0,T)$, using (3.7) and that $dx = \sqrt{t+1}d\xi$, we end up with an estimate equivalent to (2.15).

To establish (2.16), we multiply (3.8) by $(t+1)$. By the Cauchy-Schwarz inequality and since $p(\tilde{u}+w)-p(\tilde{u})-p'(\tilde{u})w$ is of quadratic order in w, we deduce

$$\begin{aligned}[(t+1)\tilde{\eta}]_t - \tilde{\eta} + [(t+1)\tilde{q}]_x + 2(t+1)z^2 \\ \leq c(t+1)|\tilde{u}_t|w^2 + (t+1)z^2 + (t+1)|p(\tilde{u})_{xt}|^2.\end{aligned} \tag{3.15}$$

Using (3.2), (3.7) and (3.3), (3.15) yields

$$\begin{aligned}[(t+1)\tilde{\eta}]_t + [(t+1)\tilde{q}]_x + (t+1)z^2 \\ \leq c\Lambda_0 w^2 + z^2 + \alpha^{-1}w^2 + c\Lambda_0^2(t+1)^{-2}e^{-\mu\xi^2}.\end{aligned} \tag{3.16}$$

Integrating the above inequality over the strip $(-\infty,\infty)\times[0,t]$, for $t\in[0,T)$, and using (3.7) and (2.15), we arrive at an estimate equivalent to (2.16). The proof is complete.

4 Stability in L^1

This section establishes the L^1 estimate (2.17), proving Theorem 2.2. The first step is to refine the Darcy law (1.3). Recall that (1.3) derives from $(1.1)_2$ upon neglecting inertial effects, setting $v_t = 0$. A higher order approximation is obtained by using for v_t the value $-p(\tilde{u})_{xt}$, suggested by (1.3). Then $(1.1)_2$ gives, in the place of (1.3),

$$\hat{v} = -p(\tilde{u})_x + p(\tilde{u})_{xt} \,. \tag{4.1}$$

It turns out that this level of precision will be necessary for proving L^1 stability.

Replacing $\tilde{v}$ by $\hat{v}$ means that one must monitor the variable $\hat{z} = v - \hat{v}$ in the place of $z = v - \tilde{v}$. Clearly, z and $\hat{z}$ are related through $z = \hat{z} + p(\tilde{u})_{xt}$. In particular, recalling (3.3), we conclude that the estimates (2.15) and (2.16), established in the previous section for (w, z), will hold for $(w, \hat{z})$ as well. Similarly, to show (2.17) it will suffice to establish an equivalent estimate for $(w, \hat{z})$. In order to avoid rewriting (2.15), (2.16) and (2.17) for $(w, \hat{z})$, in the remainder of this section we will dispense with the function z and retain the symbol z to denote the function $\hat{z}$.

In terms of the new z, the system (2.9) becomes

$$\begin{cases} w_t - z_x = p(\tilde{u})_{xxt} \\ \\ z_t + [p(\tilde{u} + w) - p(\tilde{u})]_x + z = -p(\tilde{u})_{xtt} \,. \end{cases} \tag{4.2}$$

The advantage of using (4.2) instead of (2.9) lies in that the source terms $p(\tilde{u})_{xxt}$ and $p(\tilde{u})_{xtt}$, in the former, decay faster, as $t \to \infty$, than the source term $p(\tilde{u})_{xt}$, in the latter.

The two systems (2.9) and (4.2) share the same entropy-entropy flux pairs, satisfying (3.4). For (4.2), the analog of the entropy inequality (3.5) reads

$$\begin{aligned} \eta_t + q_x + z\eta_z \quad &\leq \eta_{\tilde{u}} \tilde{u}_t + \{ q_{\tilde{u}} - [p'(\tilde{u} + w) - p'(\tilde{u})]\eta_z \} \tilde{u}_x \\ &\qquad + \eta_w p(\tilde{u})_{xxt} - \eta_z p(\tilde{u})_{xtt}. \end{aligned} \tag{4.3}$$

Proof of Theorem 2.2. To verify (2.17), we shall need an entropy-entropy flux pair (η, q) such that $\eta(w, z; \tilde{u})$ is a convex function of (w, z) and

$$\beta(|w| + |z|) \leq \eta(w, z; \tilde{u}) \leq \beta^{-1}(|w| + |z|), \tag{4.4}$$

$$z\eta_z(w, z; \tilde{u}) \geq 0\,, \tag{4.5}$$

$$|\eta_w(w, z; \tilde{u})| \leq c, \qquad |\eta_z(w, z; \tilde{u})| \leq c, \tag{4.6}$$

$$|w\eta_z(w, z; \tilde{u})| \leq c|z|, \tag{4.7}$$

$$|\eta_{\tilde{u}}(w, z; \tilde{u})| \leq c(|w| + |z|), \qquad |q_{\tilde{u}}(w, z; \tilde{u})| \leq c|z| \tag{4.8}$$

all hold for any (w, z) in some neighborhood of the origin and any $\tilde{u}$ between u_- and u_+. Indeed, assuming that (η, q) with the above properties is at hand, we write the inequality (4.3) and use (4.5), (4.6), (4.7) and (4.8), together with the Cauchy-Schwarz inequality, to majorize the right-hand side as follows:

$$\eta_t + q_x \leq w^2 + z^2 + c|\tilde{u}_t|^2 + (t+1)z^2 + c(t+1)^{-1}|\tilde{u}_x|^2 \tag{4.9}$$
$$+ c|p(\tilde{u})_{xxt}| + c|p(\tilde{u})_{xtt}|\,.$$

Integrating (4.9) over the strip $(-\infty, \infty) \times [0, t]$, for $t \in [0, T)$, and using (2.15), (2.16), (3.1), (3.2), (3.3) and (4.4), we arrive at an estimate equivalent to (2.17).

The first step toward acquiring (η, q) with the above specifications will be to construct a preliminary entropy-entropy flux pair $(\hat{\eta}, \hat{q})$ that satisfies the conditions (4.4), (4.5), (4.6), (4.7) and (4.8), but may fail to meet the remaining requirement that $\hat{\eta}(w, z; \tilde{u})$ be convex in (w, z). The pair $(\hat{\eta}, \hat{q})$ will be defined as the solution of the Cauchy problem for the linear hyperbolic system[4] (3.4), with initial condition

$$\hat{\eta}(w, 0; \tilde{u}) = |w|, \qquad \hat{q}(w, 0; \tilde{u}) = 0, \qquad -\infty < w < \infty. \tag{4.10}$$

[4]For convenience, we modify/extend $p(u)$ outside the compact interval J in such a way that $0 < \delta < -p'(u) < \delta^{-1} < \infty$, for all $u \in (-\infty, \infty)$.

By standard theory, there exists a unique Lipschitz continuous solution $(\hat{\eta}, \hat{q})$ of (3.4), (4.10) on the $w{-}z$ plane, and it has the following structure. Let us introduce the notation

$$a(w;\tilde{u}) = \sqrt{-p'(\tilde{u}+w)}\ , \tag{4.11}$$

with derivative

$$\dot{a}(w;\tilde{u}) = \frac{-p''(\tilde{u}+w)}{2\sqrt{-p'(\tilde{u}+w)}} \tag{4.12}$$

and primitive

$$\chi(w;\tilde{u}) = \int_0^w \sqrt{-p'(\tilde{u}+\omega)}d\omega. \tag{4.13}$$

We also consider the inverse function $\psi(\cdot\,;\tilde{u})$ of $\chi(\cdot\,;\tilde{u})$. The backward and the forward characteristics

$$z = -\chi(w;\tilde{u}), \qquad z = \chi(w;\tilde{u}) \tag{4.14}$$

issuing from the origin, divide the $w{-}z$ plane into four sectors

$$\begin{cases} \Omega_N = \{(w,z) : z > 0,\ -\psi(z;\tilde{u}) < w < \psi(z;\tilde{u})\} \\ \Omega_S = \{(w,z) : z < 0,\ \psi(z;\tilde{u}) < w < -\psi(z;\tilde{u})\} \\ \Omega_E = \{(w,z) : w > 0,\ -\chi(w;\tilde{u}) < z < \chi(w;\tilde{u})\} \\ \Omega_W = \{(w,z) : w < 0,\ \chi(w;\tilde{u}) < z < -\chi(w;\tilde{u})\}\ . \end{cases} \tag{4.15}$$

The functions $(\hat{\eta}, \hat{q})$ are smooth on $\Omega_N \cup \Omega_S \cup \Omega_E \cup \Omega_W$ but their derivatives experience jump discontinuities across the characteristics (4.14). In particular, the values of $(\hat{\eta}, \hat{q})$ on $\Omega_E \cup \Omega_W$ are known explicitly:

$$(\hat{\eta}, \hat{q}) = \begin{cases} (w, -z), & \text{for } (w,z) \in \Omega_E \\ (-w, z), & \text{for } (w,z) \in \Omega_W\ . \end{cases} \tag{4.16}$$

The values of $(\hat{\eta}, \hat{q})$ are also known explicitly on $\Omega_N \cup \Omega_S$ in the special case where (1.1) is linear, i.e., $a \equiv$ constant:

$$(\hat{\eta}, \hat{q}) = \begin{cases} \left(\dfrac{z}{a},\ -aw\right)\ , & \text{for } (w,z) \in \Omega_N \\ \left(-\dfrac{z}{a},\ aw\right)\ , & \text{for } (w,z) \in \Omega_S\ . \end{cases} \tag{4.17}$$

In the general, nonlinear case, we write

$$(\hat{\eta}, \hat{q}) = \begin{cases} \left(\dfrac{z}{a(w;\tilde{u})} + g(w,z;\tilde{u}),\ -a(w;\tilde{u})w + h(w,z;\tilde{u})\right),\ (w,z) \in \Omega_N, \\ \left(-\dfrac{z}{a(w;\tilde{u})} - g(w,z;\tilde{u}),\ a(w;\tilde{u})w - h(w,z;\tilde{u})\right),\ (w,z) \in \Omega_S\ . \end{cases} \tag{4.18}$$

Note that $(\hat{\eta}, \hat{q})$ defined by (4.16), (4.17) satisfy the requisite conditions (4.4), (4.5), (4.6), (4.7) and (4.8). It follows that the modified $(\hat{\eta}, \hat{q})$ defined by (4.16), (4.18) will equally satisfy these conditions near the origin, so long as the "correction" (g, h) is of quadratic order, namely (a) g and h are $O(w^2 + z^2)$; (b) the first derivatives $g_w, g_z, g_{\tilde{u}}, h_w, h_z, h_{\tilde{u}}$ are $O(|w| + |z|)$; and (c) the second derivatives $g_{ww}, g_{wz}, g_{zz}, h_{ww}, h_{wz}, h_{zz}$ are $O(1)$. We proceed to show that this is indeed the case. Since $g(-w,-z;\tilde{u}) = g(w,z;\tilde{u})$ and $h(-w,-z;\tilde{u}) = h(w,z;\tilde{u})$, it will suffice to work in Ω_N.

By virtue of (4.18) and (3.4), (g, h) satisfies the system

$$\begin{cases} g_z = -\dfrac{h_w}{a^2} + \dfrac{\dot{a}}{a^2}w \\ h_z = -g_w + \dfrac{\dot{a}}{a^2}z. \end{cases} \tag{4.19}$$

Furthermore, since $(\hat{\eta}, \hat{q})$ is continuous across the characteristics (4.14), upon combining (4.16) with (4.18) we deduce

(4.20)
$$\begin{cases} g(w,\chi(w;\tilde u);\tilde u) = w - \dfrac{\chi(w;\tilde u)}{a(w;\tilde u)}\,, \;\; h(w,\chi(w;\tilde u);\tilde u) = -\chi(w;\tilde u) + a(w;\tilde u)w, \\[2ex] g(w,-\chi(w;\tilde u);\tilde u) = -w + \dfrac{\chi(w;\tilde u)}{a(w;\tilde u)}\,, \;\; h(w,-\chi(w;\tilde u);\tilde u) = -\chi(w;\tilde u) + a(w;\tilde u)w. \end{cases}$$

At this point, it is convenient to perform a coordinate transformation, from (w,z) to Riemann invariants (ζ,ξ):

$$\begin{cases} \zeta = \zeta(w,z;\tilde u) = z - \chi(w;\tilde u), \\[2ex] \xi = \xi(w,z;\tilde u) = z + \chi(w;\tilde u), \end{cases} \tag{4.21}$$

which maps Ω_N onto the first quadrant $\{(\zeta,\xi) : \zeta > 0, \xi > 0\}$. At the same time, we pass from (g,h) to new variables (r,s):

$$r = g - \frac{h}{a}, \qquad s = g + \frac{h}{a}\,. \tag{4.22}$$

In the new coordinate system and new variables, (4.19) and (4.20) become

$$\begin{cases} r_\zeta = \dfrac{\dot a}{4a^2}(r-s) + \dfrac{\dot a}{2a^2}w - \dfrac{\dot a}{2a^3}z, \\[2ex] s_\xi = \dfrac{\dot a}{4a^2}(r-s) + \dfrac{\dot a}{2a^2}w + \dfrac{\dot a}{2a^3}z, \end{cases} \tag{4.23}$$

$$\begin{cases} r(0,\xi;\tilde u) = 0, & \text{for } \xi > 0 \\[2ex] s(\zeta,0;\tilde u) = 0, & \text{for } \zeta > 0. \end{cases} \tag{4.24}$$

The coefficients on the right-hand side of (4.23) are realized as functions of $(\zeta,\xi;\tilde u)$ through the inverse transformation to (4.21), which reads $w = \psi(\frac{1}{2}(\xi-\zeta);\tilde u), z = \frac{1}{2}(\xi+\zeta)$.

Fix $\epsilon > 0$ small and suppose $|r| + |s|$ attains its maximum, say m, on the square $[0,\epsilon]\times[0,\epsilon]$ at a point $(\bar\zeta,\bar\xi)$. Integrating $(4.23)_1$ with respect to

ζ on $[0,\bar{\zeta}]$, keeping ξ fixed, equal to $\bar{\xi}$, then integrating $(4.23)_2$ with respect to ξ on $[0,\bar{\xi}]$, keeping ζ fixed, equal to $\bar{\zeta}$, and using (4.24), we deduce an inequality of the form $m \le c\epsilon m + c\epsilon^2$, whence it follows that both r and s are $O(\zeta^2+\xi^2)$ near the origin. In turn, (4.23) implies that r_ζ and s_ξ are $O(|\zeta|+|\xi|)$ near the origin.

To get bounds for $r_{\tilde{u}}$ and $s_{\tilde{u}}$, we note that when w and z are regarded as functions of $(\zeta,\xi;\tilde{u})$, then $w_{\tilde{u}}=-1, z_{\tilde{u}}=0$, and thereby $a_{\tilde{u}}=0, \dot{a}_{\tilde{u}}=0$. Therefore, differentiating (4.23) with respect to $\tilde{u}$ yields

$$(4.25)\qquad \begin{cases} r_{\tilde{u}\zeta} = \dfrac{\dot{a}}{4a^2}(r_{\tilde{u}}-s_{\tilde{u}}) - \dfrac{\dot{a}}{2a^2}, \\[2ex] s_{\tilde{u}\xi} = \dfrac{\dot{a}}{4a^2}(r_{\tilde{u}}-s_{\tilde{u}}) - \dfrac{\dot{a}}{2a^2}\,. \end{cases}$$

Moreover, (4.24) imply $r_{\tilde{u}}(0,\xi;\tilde{u})=0$, for $\xi>0$, and $s_{\tilde{u}}(\zeta,0;\tilde{u})=0$, for $\zeta>0$. Thus, the argument used above for bounding r and s applies here as well and yields that $r_{\tilde{u}}$ and $s_{\tilde{u}}$ are $O(|\zeta|+|\xi|)$ near the origin.

Next we differentiate $(4.23)_1$ with respect to ξ and $(4.23)_2$ with respect to ζ. Since $w_\zeta=-1/2a, w_\xi=1/2a, z_\zeta=1/2$ and $z_\xi=1/2$, we deduce

$$(4.26)\qquad \begin{cases} r_{\xi\zeta} - \dfrac{\dot{a}}{4a^2}r_\xi = O(1), \\[2ex] s_{\zeta\xi} + \dfrac{\dot{a}}{4a^2}s_\zeta = O(1). \end{cases}$$

Integrating the above ordinary differential equations for r_ξ and s_ζ, starting out from initial conditions $r_\xi(0,\xi;\tilde{u})=0$ and $s_\zeta(\zeta,0;\tilde{u})=0$, which follow from (4.24), we deduce that r_ξ and s_ζ are $O(|\zeta|+|\xi|)$ near the origin. In turn, (4.26) implies that $r_{\xi\zeta}$ and $s_{\zeta\xi}$ are $O(1)$ near the origin.

Differentiating $(4.23)_1$ with respect to ζ and $(4.23)_2$ with respect to ξ, we deduce that $r_{\zeta\zeta}$ and $s_{\xi\xi}$ are $O(1)$ near the origin. To estimate the remaining second derivatives $r_{\xi\xi}$ and $s_{\zeta\zeta}$, we differentiate $(4.23)_1$ twice with respect to

ξ and $(4.23)_2$ twice with respect to ζ, which yields

$$
(4.27) \qquad \begin{cases} r_{\xi\xi\zeta} - \dfrac{\dot{a}}{4a^2} r_{\xi\xi} = O(1), \\[2ex] s_{\zeta\zeta\xi} + \dfrac{\dot{a}}{4a^2} s_{\zeta\zeta} = O(1). \end{cases}
$$

Since $r_{\xi\xi}(0,\xi;\tilde{u}) = 0$ and $s_{\zeta\zeta}(\zeta,0;\tilde{u}) = 0$, integrating the above ordinary differential equations, we conclude that $r_{\xi\xi}$ and $s_{\zeta\zeta}$ are $O(|\zeta| + |\xi|)$ near the origin.

We have thus verified that the entropy-entropy flux pair $(\hat{\eta}, \hat{q})$, defined by (4.16), (4.18) satisfies the requirements (4.4), (4.5), (4.6), (4.7) and (4.8). However, $\hat{\eta}(w, z; \tilde{u})$ just misses being convex near the origin, for two reasons: The characteristics (4.14) are curved; the Hessian of the (small) perturbation (g, h) is bounded but not necessarily positive in the vicinity of $(0, 0)$. On the other hand, notice that the entropy-entropy flux pair $(\tilde{\eta}, \tilde{q})$ defined by (3.6) satisfies the convexity condition as well as (4.5), (4.6), (4.7) and (4.8), but fails to meet the requirement (4.4), as $\tilde{\eta}$ is of quadratic order at the origin. We may thus produce an entropy-entropy flux pair (η, q) with all the desired properties by combining $(\hat{\eta}, \hat{q})$ with $(\tilde{\eta}, \tilde{q})$:

$$
(4.28) \qquad \begin{cases} \eta(w, z; \tilde{u}) = \hat{\eta}(w, z; \tilde{u}) + \kappa \tilde{\eta}(w, z; \tilde{u}), \\[2ex] q(w, z; \tilde{u}) = \hat{q}(w, z; \tilde{u}) + \kappa \tilde{q}(w, z; \tilde{u}), \end{cases}
$$

where κ is a positive constant. Indeed, this (η, q) satisfies (4.4), (4.5), (4.6), (4.7) and (4.8). Furthermore, if κ is sufficiently large, $\eta(w, z; \tilde{u})$ is convex in (w, z).

As noted above, once (η, q) with the required specifications is available, integration of (4.9) yields (2.17). The proof is complete.

5 Global BV Solutions

This section establishes the existence of globally defined, admissible BV solutions to the Cauchy problem (2.9), (2.10), and thereby to the Cauchy problem

(1.1), (1.2), by proving Theorem 2.3. The analysis will rely on the theory of hyperbolic systems of balance laws with inhomogeneity and dissipation, and will make essential use of a change of variable that redistributes damping between the two equations in (2.9).

We consider a system of balance laws in the form

$$U_t + F(U,x,t)_x + AU = G(x,t). \tag{5.1}$$

The state vector U takes values in a ball $\mathcal{B}$ of $I\!R^n$, which is centered at 0 and has radius ρ. The flux F is a smooth function defined on $\mathcal{B}\times(-\infty,\infty)\times[0,\infty)$ and taking values in $I\!R^n$. Its partial derivatives of first and second order are uniformly bounded. For fixed $U \in \mathcal{B}$, $x \in (-\infty,\infty)$, $t \in [0,\infty)$, the Jacobian matrix (with respect to the U-variable) $DF(U,x,t)$ has real distinct eigenvalues that are uniformly separated. Thus, (5.1) is strictly hyperbolic. A is a constant $n \times n$ matrix. The source G is a uniformly Lipschitz function on $(-\infty,\infty)\times[0,\infty)$ taking values in $I\!R^n$ and $G(\pm\infty,t) = 0$. The intensity of the inhomogeneity and the strength of the source will be controlled through a positive parameter a, by imposing the conditions

$$F_x(0,x,t) = 0, \qquad |DF_x(U,x,t)| \le a, \qquad |DF_t(U,x,t)| \le a, \tag{5.2}$$

$$\int_{-\infty}^{\infty} |DF_x(U,x,t)|dx \le a, \ \int_{-\infty}^{\infty} |F_{xt}(U,x,t)|dx \le a, \ \int_{-\infty}^{\infty} |G_x(x,t)|dx \le a, \tag{5.3}$$

for all $U \in \mathcal{B}$, $x \in (-\infty,\infty)$, $t \in [0,\infty)$.

We prescribe initial data

$$U(x,0) = U_0(x), \qquad -\infty < x < \infty, \tag{5.4}$$

that vanish at $\pm\infty$ and have bounded variation over $(-\infty,\infty)$. By combining the random choice algorithm [15], which solves the Cauchy problem for homogeneous systems of conservation laws, with operator splitting, in order to account for the effects of inhomogeneity and source, the following local existence theorem[5] for solutions to the Cauchy problem (5.1), (5.4) is established in [11]:

Proposition 5.1. *There are positive constants* a_0, V_0 *and* T *such that if* (5.2), (5.3) *hold for* $a \leq a_0$ *and* $TVU_0(\cdot) \leq V_0$, *then there exists an admissible* BV *solution* U *of the Cauchy problem* (5.1), (5.4) *defined on the strip* $(-\infty, \infty) \times [0, T)$.

It is further shown in [11] that when the matrix A in (5.1) satisfies a certain dissipativeness condition, then the total variation of the solution U in Proposition 5.1 is bounded independently of T. This may allow extending U into a global solution, defined on the entire upper half-plane. The following special case where A is a multiple of the identity matrix I will suffice for the present purposes:

Proposition 5.2. *Suppose that in* (5.1) $A = \frac{1}{2}I$. *Then, under the assumptions of Proposition* 5.1, *the solution* U *of* (5.1), (5.4) *satisfies the estimate*

$$TVU(\cdot, t) \leq Ke^{-\nu t} TVU_0(\cdot) + Ka, \tag{5.5}$$

for any $t \in [0, T)$, *where* K *and* ν *are positive constants independent of* T.

Uniqueness of admissible BV solutions for the Cauchy problem (5.1), (5.4) has been established in [2,7] in the homogeneous case, where F and G do not depend explicitly on (x, t). Extending this analysis to the inhomogeneous case appears to be a doable, though tedious, task. Accordingly, we shall operate here under the assumption that when $A = \frac{1}{2}I$ then, irrespective of the way it was constructed, any BV solution of (5.1), (5.4) whose local structure is compatible with the admissible solution of the Riemann problem is unique and thus satisfies the estimate (5.5).

We have now laid the groundwork for the

[5]The slightly more stringent regularity and normalization conditions imposed on the source in [11] may be relaxed to the present assumptions, without any difficulty.

Proof of Theorem 2.3. The system (2.9) is of the form (5.1). By virtue of (3.1), (3.2) and (3.3), the conditions (5.2) and (5.3) are satisfied with $a = c\Lambda_0$. Thus, when Λ_0 and M are sufficiently small, Proposition 5.1 implies that there exists an admissible BV solution of the Cauchy problem (2.9), (2.10), defined on a strip $(-\infty, \infty) \times [0, T)$.

Since we are seeking a global solution to (2.9), (2.10), we need a uniform bound on the total variation of the local solution. It is not possible to achieve that by appealing directly to Proposition 5.2, because the matrix A associated with (2.9) does not satisfy the dissipativeness condition, as the damping is inequitably distributed between the two equations. We shall overcome this difficulty by redistributing the damping via a device originally employed in [8], for the same purpose, in the context of the special case $u_- - u_+$. We retain w but replace z by the new variable

$$\omega = z + \tfrac{1}{2}\phi, \tag{5.6}$$

where ϕ is the potential defined by (2.13). (This transformation was first introduced by Feireisl [14], in an unrelated context). Recalling (2.14), we deduce that (2.9) is written in terms of the new variables as

$$\begin{cases} w_t - \omega_x + \frac{1}{2}w = 0 \\ \\ \omega_t + [p(\tilde{u} + w) - p(\tilde{u})]_x + \frac{1}{2}\omega = \frac{1}{4}\phi + p(\tilde{u})_{xt} \,. \end{cases} \tag{5.7}$$

We notice that the damping is now equidistributed between the two equations. Indeed, (5.7) is a system of the form (5.1), with $A = \frac{1}{2}I$, as required in Proposition 5.2. The potential ϕ appearing on the right-hand side of $(5.7)_2$ is a nonlocal functional of the solution (w, ω), and thus a priori unknown, but it shall be regarded here as part of the source.

As noted above, we already have an admissible BV solution (w, z) of (2.9), (2.10) on $(-\infty, \infty) \times [0, T)$, and this induces an admissible BV solution (w, ω) of (5.7), on the same strip, with initial conditions

$$w(x, 0) = w_0(x)\,, \quad \omega(x, 0) = \omega_0(x) = z_0(x) + \tfrac{1}{2}\phi_0(x), \quad -\infty < x < \infty. \tag{5.8}$$

Since $\phi_x = w$, (2.17) together with (3.3) imply

$$\int_{-\infty}^{\infty} |\tfrac{1}{4}\phi_x(x,t) + p(\tilde{u}(x,t))_{xxt}|dx \leq c\Lambda, \tag{5.9}$$

for all $t \in [0, T)$. Thus, when Λ is sufficiently small to render the right-hand side of (5.9) less than a_0, Proposition 5.2 applies and the estimate (5.5) yields

$$TVw(\cdot,t) + TV\omega(\cdot,t) \leq Ke^{-\nu t}[TVw_0(\cdot) + TV\omega_0(\cdot)] + c\Lambda, \tag{5.10}$$

for all $t \in [0, T)$. In turn, (5.10) together with (2.17), (3.3), (5.8) and (2.7) give

$$TVw(\cdot,t) + TVz(\cdot,t) \leq KMe^{-\nu t} + c\Lambda, \tag{5.11}$$

for all $t \in [0, T)$. When Λ and M are sufficiently small so that $KM + c\Lambda < V_0$, we may apply Proposition 5.1 once again, starting at $t = T$, and extend the solution (w, z) of (2.9), (2.10) to the strip $(-\infty, \infty) \times [0, 2T)$. Furthermore, (5.11) will now hold for all $t \in [0, 2T)$. By repeating this process, we establish the existence of an admissible BV solution to (2.9), (2.10) that is defined on the entire upper half-plane and satisfies an estimate equivalent to (2.19), for all $t \in [0, \infty)$. This completes the proof.

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