BIAS REDUCTION FOR ENDPOINT ESTIMATION

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Abstract: Recently Li and Peng (2009a) proposed a bias reduction method for estimating the endpoint of a distribution function via an external estimator for the so-called second order parameter. Unlike the same study for the tail index of a heavy tailed distribution, the above procedure requires a certain rate of convergence of the external estimator rather than consistence. This makes the choice of such an external estimator impractical. In this paper, we propose a new bias reduction method which estimates all parameters by using the same number of upper order statistics.

Key words and phrases: Bias reduction, endpoint, extreme value index, order statistics.

1 Introduction

Suppose $X_1, \cdots, X_n$ are independent and identically distributed random variables with distribution function $F(x)$. Let θ denote the right endpoint of F, i.e., $\theta = \sup\{x : F(x) < 1\}$. When θ is finite, both point and interval estimation for θ has been studied in the literature, see Athreya and Fukuchi (1997), Hall (1982), Hall and Wang (1999), Loh (1984), Smith (1987) and Woodroofe (1974). Specifically, by assuming that

$$1 - F(x) = c(\theta - x)^\alpha \{1 + O((\theta - x)^\beta)\} \tag{1.1}$$

as $x \to \theta$, where $c > 0, \alpha > 2, \beta > 0$, Hall (1982) studied the following estimator

$$\begin{aligned} & (\hat{c}_{Hall}(k), \hat{\theta}_{Hall}(k), \hat{\alpha}_{Hall}(k)) \\ = \; & \arg\max_{(c,\theta,\alpha)} \tfrac{n!}{(n-k)!} \Pi_{i=0}^{k-1} \{c\alpha(\theta - Z_i)^{\alpha-1}\}\{1 - c(\theta - Z_k)^\alpha\}^{n-k}, \end{aligned} \tag{1.2}$$

where $Z_i = X_{n,n-i}$, $X_{n,1} \le \cdots \le X_{n,n}$ denote the order statistics of $X_1, \cdots, X_n$ and $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$. Moreover, Hall (1982) derived the asymptotic limit of $(\hat{\theta}_{Hall}(k), \hat{\alpha}_{Hall}(k))$ when $\sqrt{k}(k/n)^{\beta/\alpha} \to 0$. In order to choose k in terms of the asymptotic mean squared errors, Li and Peng (2009b) derived the asymptotic limit of $(\hat{\theta}_{Hall}(k), \hat{\alpha}_{Hall}(k))$ when $\sqrt{k}(k/n)^{\beta/\alpha} \to \lambda \in [0, \infty)$ and

$$1 - F(x) = c(\theta - x)^{\alpha} + d(\theta - x)^{\alpha+\beta} + o((\theta - x)^{\alpha+\beta}) \tag{1.3}$$

as $x \to \theta$. Some applications of endpoint estimation can be found in De Haan and Ferreira (2006).

Motivated by the recent studies on bias reduction estimation for a tail index, Li and Peng (2009a) proposed a bias reduction method for the endpoint as follows. Write

$$\begin{aligned} & L(\theta, \alpha, c, d, \beta) \\ = \quad & \tfrac{n!}{(n-k)!}\Pi_{i=0}^{k-1}\{c\alpha(\theta - Z_i)^{\alpha-1} + d(\alpha+\beta)(\theta - Z_i)^{\alpha+\beta-1}\}\times \\ & \{1 - c(\theta - Z_k)^{\alpha} - d(\theta - Z_k)^{\alpha+\beta}\}^{n-k}. \end{aligned} \tag{1.4}$$

For a given β, define

$$(\hat{\theta}(k;\beta), \hat{\alpha}(k;\beta), \hat{c}(k;\beta), \hat{d}(k;\beta)) = \arg \max_{(\theta,\alpha,c,d)} L(\theta, \alpha, c, d, \beta). \tag{1.5}$$

Hence, the bias reduction estimators proposed in Li and Peng (2009a) are $\hat{\theta}(k; \hat{\beta}(k_1))$ and $\hat{\alpha}(k; \hat{\beta}(k_1))$ where $\hat{\beta}(k_1)$ is an external estimator of β by using the upper k_1 order statistics.

Let $U(t)$ denote the inverse function of $1/(1-F(t))$. Assume that there exist functions $a(t) > 0$, $A(t)$ and $B(t)$ such that

$$\lim_{t\to\infty} B^{-1}(t)\Big\{\frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^{\gamma}-1}{\gamma}}{A(t)} - H_{\gamma,\rho}(x)\Big\} = R_{\gamma,\rho,\eta}(x) \tag{1.6}$$

for some $\eta < 0$, where $\gamma = -1/\alpha \in (-1/2, 0)$, $\rho = -\beta/\alpha < 0$,

$$H_{\gamma,\rho}(x) = \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1}\, dudy$$

and

$$R_{\gamma,\rho,\eta}(x) = \int_1^x y^{\gamma-1} \int_1^y u^{\rho-1} \int_1^u s^{\eta-1}\, dsdudy.$$

Let $(\theta_0, \alpha_0, \beta_0)$ denote the true value of (θ, α, β). Specifically Li and Peng (2009a) derived the asymptotic limit of $(\hat{\theta}(k; \hat{\beta}(k_1)), \hat{\alpha}(k; \hat{\beta}(k_1)))$ when k satisfies

$$k^{1/2}|A(n/k)| \to \infty, \quad k^{1/2}A^2(n/k) \to 0, \quad k^{1/2}A(n/k)B(n/k) \to 0 \tag{1.7}$$

and $\hat{\beta}(k_1) - \beta_0 = o_p(\frac{1}{k^{1/2}A(n/k)})$. Hence, the above condition on the rate of convergence makes the choice of the external estimator for β quite impractical. This is in contrast to the study of applying the same bias reduction technique to tail index estimation, where the condition $\sqrt{k}A(n/k) \to \lambda \in (-\infty, \infty)$ is required instead of $\sqrt{k}|A(n/k)| \to \infty$ and a consistent estimator for the second order parameter is enough; see Careiro, Figueiredo and Gomes (2004) and Gomes and Martins (2002, 2004).

In order to avoid the difficult choice of an external estimator for β, in this paper we propose to employ the bias reduction technique for a tail index in Feuerverger and Hall (1999) and Peng and Qi (2004) to the endpoint estimation. That is, we propose to maximize $L(\theta, \alpha, c, d, \beta)$ over these five parameters simultaneously.

We organize this paper as follows. Main results are given in Section 2. A simulation study is provided in Section 3. All proofs are put in Section 4.

2 Main results

Note that condition (1.6) implies (1.3). Let's put $\beta^* = \alpha + \beta$,

$$\gamma = (\gamma_1, \cdots, \gamma_5)^T = (\theta, \alpha, \beta^*, c, d)^T \quad \text{and} \quad L_1(\gamma) = L(\theta, \alpha, c, d, \beta).$$

First we show that there is a consistent estimator as follows.

Theorem 2.1. *Suppose conditions (1.6) and (1.7) hold. Then there exist solutions, say $\hat{\gamma}(k) = (\hat{\theta}(k), \hat{\alpha}(k), \hat{\beta}^*(k), \hat{c}(k), \hat{d}(k))^T$, of the score equations $\frac{\partial}{\partial \gamma} L_1(\gamma) = 0$ such that*

$$\hat{\alpha}(k) \xrightarrow{p} \alpha_0, \quad \hat{\beta}^*(k) \xrightarrow{p} \beta_0^*, \quad \hat{c}(k) \xrightarrow{p} c, \quad \hat{d}(k) \xrightarrow{p} d, \quad (\hat{\theta}(k) - \theta_0)/a(n/k) \xrightarrow{p} 0,$$

where $(\theta_0, \alpha_0, \beta_0^*, c_0, d_0)^T$ denotes the true value of γ.

By simplifying the score equations, we can show that the above estimator $(\hat{\theta}(k), \hat{\alpha}(k), \hat{\beta}^*(k))^T$ is indeed a solution to

$$\begin{cases} \int_0^1 Q_{1n}(t; \theta, \alpha, \beta^*) dt = 0 \\ \int_0^1 Q_{2n}(t; \theta, \alpha, \beta^*) dt = 0 \\ \int_0^1 Q_{3n}(t; \theta, \alpha, \beta^*) dt = 0, \end{cases} \tag{2.1}$$

where

$$Q_{1n}(t;\theta,\alpha,\beta^*) = \frac{(\alpha T_n - \frac{\alpha}{\beta^*}) + (1-\alpha T_n)(\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^*-\alpha}}{\alpha(\alpha T_n - \frac{\alpha}{\beta^*}) + \beta^*(1-\alpha T_n)(\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^*-\alpha}} + \log\Big(\frac{\theta - Z_{[kt]}}{\theta - Z_k}\Big), \tag{2.2}$$

$$Q_{2n}(t;\theta,\alpha,\beta^*) = \frac{\alpha(\alpha-1)(\alpha T_n - \frac{\alpha}{\beta^*}) + \beta^*(\beta^*-1)(1-\alpha T_n)(\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^*-\alpha}}{(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\{\alpha(\alpha T_n - \frac{\alpha}{\beta^*}) + \beta^*(1-\alpha T_n)(\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^*-\alpha}\}} - \{\alpha + \beta^* - \alpha\beta^* T_n\}, \tag{2.3}$$

$$Q_{3n}(t;\theta,\alpha,\beta^*) = \frac{\alpha(\beta^*-\alpha)}{\alpha\beta^*(\alpha T_n - \frac{\alpha}{\beta^*}) + (\beta^*)^2(1-\alpha T_n)(\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^*-\alpha}} - 1 \tag{2.4}$$

and

$$T_n = \int_0^1 \log\Big(\frac{\theta - Z_k}{\theta - Z_{[kt]}}\Big)dt \xrightarrow{p} \int_0^1 \log(t^{-1/\alpha})dt = 1/\alpha;$$

see Section 3 for a detailed derivation.

The following theorem gives the asymptotic limit of the new bias reduction estimator.

Theorem 2.2. *Suppose conditions (1.6) and (1.7) hold. Then there exists a Brownian motion W such that*

$$\Big(\sqrt{k}(\hat{\alpha}(k)-\alpha_0),\ \sqrt{k}A(n/k)(\hat{\beta}^*(k)-\beta_0^*),\ \sqrt{k}(\frac{\hat{\theta}(k)-\theta_0}{a(n/k)})\Big)^T \xrightarrow{d} \Sigma^{-1}\big(N_1,\ N_2,\ N_3\big)^T,$$

where

$$\begin{cases} N_1 &= \frac{\beta_0^*(\beta_0^*-\alpha_0)}{\alpha_0^2}\int_0^1 \{\frac{1}{\alpha_0}t^{\beta_0^*/\alpha_0-2} - \frac{1}{2\beta_0^*-\alpha_0}t^{-1}\}\{W(t)-tW(1)\}dt \\ N_2 &= \int_0^1 \{(1-\frac{1}{\alpha_0})t^{-1/\alpha_0-1} - \frac{\beta_0^*}{\beta_0^*-1}t^{-1}\}\{W(t)-tW(1)\}dt \\ N_3 &= \frac{(\beta_0^*)^2}{\alpha_0^2}\int_0^1 t^{\beta_0^*/\alpha_0-2}\{W(t)-tW(1)\}dt, \end{cases}$$

and $\Sigma = (\sigma_{ij})$ *with*

$$\begin{cases} \sigma_{11} &= \frac{\beta_0^*(\beta_0^*-\alpha_0)}{\alpha_0^3(2\beta_0^*-\alpha_0)} + \frac{\beta_0^*-\alpha_0}{\alpha_0^2\beta_0^*}, \\ \sigma_{12} &= \frac{(\beta_0^*)^2+2\alpha_0\beta_0^*-\alpha_0^2}{\beta_0^*(2\beta_0^*-\alpha_0)^2(\alpha_0-\beta_0^*-1)}, \\ \sigma_{13} &= \frac{\alpha_0-\beta_0^*}{\alpha_0^2(\beta_0^*-1)} + \frac{\beta_0^*(\alpha_0-\beta_0^*)}{\alpha_0^2(\alpha_0-1)(2\beta_0^*-\alpha_0)}, \\ \sigma_{21} &= \frac{\beta_0^*-\alpha_0}{\alpha_0(\alpha_0-1)(\beta_0^*-1)}, \\ \sigma_{22} &= \frac{\alpha_0^2}{(\alpha_0-\beta_0^*-1)\beta_0^*(\beta_0^*-1)^2}, \\ \sigma_{23} &= \frac{\alpha_0-\beta_0^*-1}{(\beta_0^*-1)(\alpha_0-1)(\alpha_0-2)} \\ \sigma_{31} &= \frac{1}{\alpha_0} + \frac{(\beta_0^*)^2}{\alpha_0^2(2\beta_0^*-\alpha_0)}, \\ \sigma_{32} &= \frac{\alpha_0(\beta_0^*-\alpha_0)}{(\alpha_0-\beta_0^*-1)(2\beta_0^*-\alpha_0)^2}, \\ \sigma_{33} &= -\frac{(\beta_0^*)^2}{\alpha_0(\alpha_0-1)(2\beta_0^*-\alpha_0)} - \frac{\beta_0^*}{\alpha_0(\beta_0^*-1)}. \end{cases}$$

Remark 2.1. *Since* $\sigma_{11}\frac{\alpha_0\beta_0^*}{\beta_0^*-\alpha_0} = \sigma_{31}$, $\sigma_{13}\frac{\alpha_0\beta_0^*}{\beta_0^*-\alpha_0} = \sigma_{33}$ *and* $\sigma_{12}\frac{\alpha_0\beta_0^*}{\beta_0^*-\alpha_0} \neq \sigma_{32}$, *we only need to show that* $\sigma_{21}\sigma_{33} - \sigma_{23}\sigma_{31} \neq 0$ *in order to show that* $|\Sigma| \neq 0$. *A straightforward calculation shows that*

$$\begin{aligned} & \{\sigma_{21}\sigma_{33} - \sigma_{23}\sigma_{31}\}\alpha_0(\alpha_0-1)(\beta_0^*-1) \\ = \; & -\frac{(\beta_0^*)^2(\beta_0^*-\alpha_0)}{\alpha_0(\alpha_0-1)(2\beta_0^*-\alpha_0)} - \frac{\beta_0^*(\beta_0^*-\alpha_0)}{\alpha_)(\beta_0^*-1)} + \frac{\beta_0^*-\alpha_0+1}{\alpha_0-2} + \frac{(\beta_0^*)^2(\beta_0^*-\alpha_0+1)}{\alpha_0(\alpha_0-2)(2\beta_0^*-\alpha_0)} \\ = \; & \frac{(\beta_0^*-\alpha_0+1)\alpha_0(\beta_0^*-1)-(\alpha_0-2)\beta_0^*(\beta_0^*-\alpha_0)}{\alpha(\alpha-2)(\beta_0^*-1)} \\ & + \frac{(\beta_0^*)^2\{(\beta_0^*-\alpha_0+1)(\alpha_0-1)-(\beta_0^*-\alpha_0)(\alpha_0-2)\}}{\alpha_0(\alpha_0-1)(\alpha_0-2)(2\beta_0^*-\alpha_0)} \\ = \; & \frac{\alpha_0(\beta_0^*-1)+(\beta_0^*-\alpha_0)(2\beta_0^*-\alpha_0)}{\alpha_0(\alpha_0-2)(\beta_0^*-1)} + \frac{(\beta_0^*)^2(\beta_0^*-1)}{\alpha_0(\alpha_0-1)(\alpha_0-2)(2\beta_0^*-\alpha_0)} \\ > \; & 0 \end{aligned}$$

since $\alpha_0 > 2$ *and* $\beta_0^* > \alpha_0$. *Hence the* Σ *defined in Theorem 2.2 is invertible.*

Remark 2.2. *It is difficult to theoretically compare the above bias reduction estimators with the bias reduction estimators in Li and Peng (2008) due to the complicated formulas of the asymptotic variances. The simulation study given in the next section indicates that the above bias reduction estimators have slightly larger mean squared errors than the corresponding ones in Li and Peng (2009a) with true second order parameter as the external estimator. As mentioned in the introduction, the external estimator for* β *in Li and Peng (2009a) requires a certain rate of convergence, which is hard to obtain. Thus, the new bias reduction estimators are practically favorable than the ones in Li and Peng (2009a).*

3 Simulation

In this section, we compare the proposed bias reduction estimators, which estimate five parameters simultaneously, with the bias reduction estimator in Li and Peng (2009a), where the external estimator of β is chosen as the true value. We denote these two estimators by *Esti. beta* and *True beta* in Figures 1-4 below. Note that the simulation study in Li and Peng (2009a) employed a simple consistent estimator for β, which does not achieve the required rate of convergence in general.

We drew 1,000 random samples of size $n = 1000$ and 3000 from a random variable $X = \theta - 1/Y$, where Y has a Burr distribution $P(Y > y) = (1 + y^{\tau_1})^{-\tau_2}$ for some $\tau_1, \tau_2 > 0$. The distribution of X satisfies (1.3) with $\alpha = \tau_1\tau_2$ and $\beta = \tau_1$. We consider the cases $\theta = 0$, $(\tau_1, \tau_2, \theta) = (2, 2, 0)$ and $(1, 4, 0)$. For each case, we compute these estimators by increasing k from 50 to 800 with step 10 in case of $n = 1000$, and from 50 to 2000 with step 25 in case of $n = 3000$. When there is no solution to either method, we replace the estimators for θ and α by the sample maximum and $\max\{2, -1/\hat{\gamma}^M(k)\}$, respectively, where $\hat{\gamma}^M(k)$ is the so-called moment estimator for the extreme value index in Dekkers, Einmahl and de Haan (1989). The means and root of mean squared errors (RMSE) of the estimators for α and θ are plotted in Figures 1–4. From these figures, the new bias reduction estimators have a slightly larger RMSE than the ones in Li and Peng (2009a). However, the new estimators avoid the difficult choice of an external estimator for β.

4 Proofs

Derivation of (2.1). Score equations are:

$$\begin{aligned}\frac{\partial \log L_1(\theta, \alpha, \beta^*, c, d)}{\partial c} &= \sum_{i=0}^{k-1} \frac{\alpha(\theta - Z_i)^{\alpha-1}}{c\alpha(\theta - Z_i)^{\alpha-1} + d\beta^*(\theta - Z_i)^{\beta^*-1}} \\ &\quad - \frac{(n-k)(\theta - Z_k)^{\alpha}}{1 - c(\theta - Z_k)^{\alpha} - d(\theta - Z_k)^{\beta^*}} = 0,\end{aligned} \tag{4.1}$$

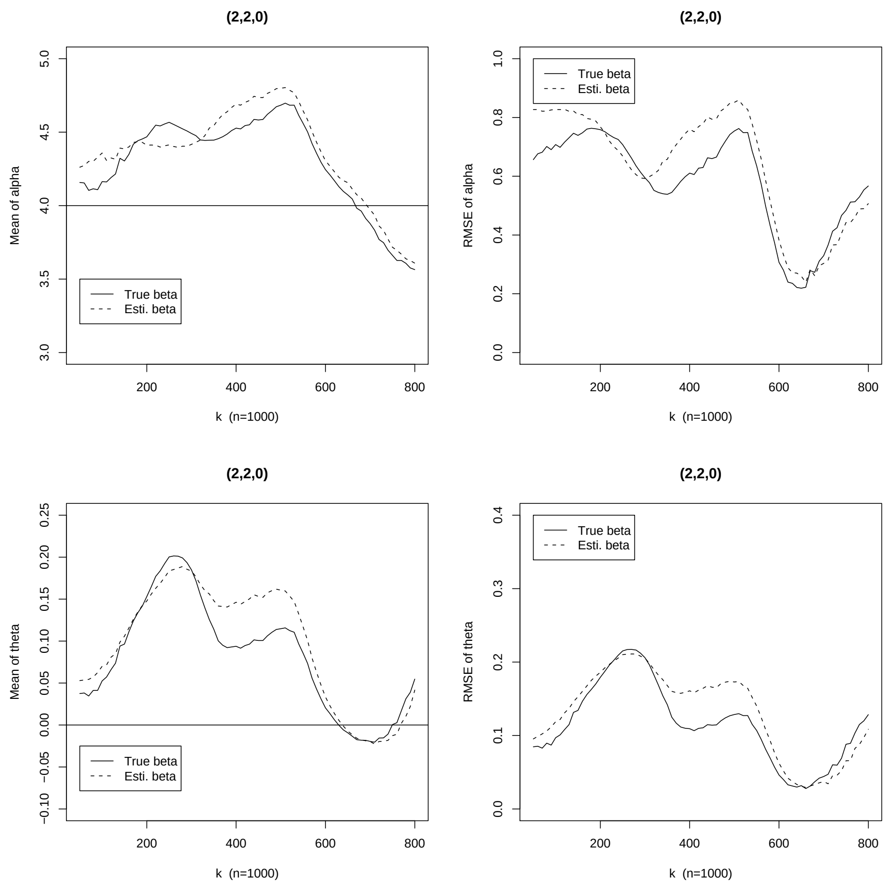

Figure 1: Mean and root of mean squared error (RMSE) are plotted against different k for the case of $(\tau_1, \tau_2, \theta) = (2, 2, 0)$ and $n = 1000$.

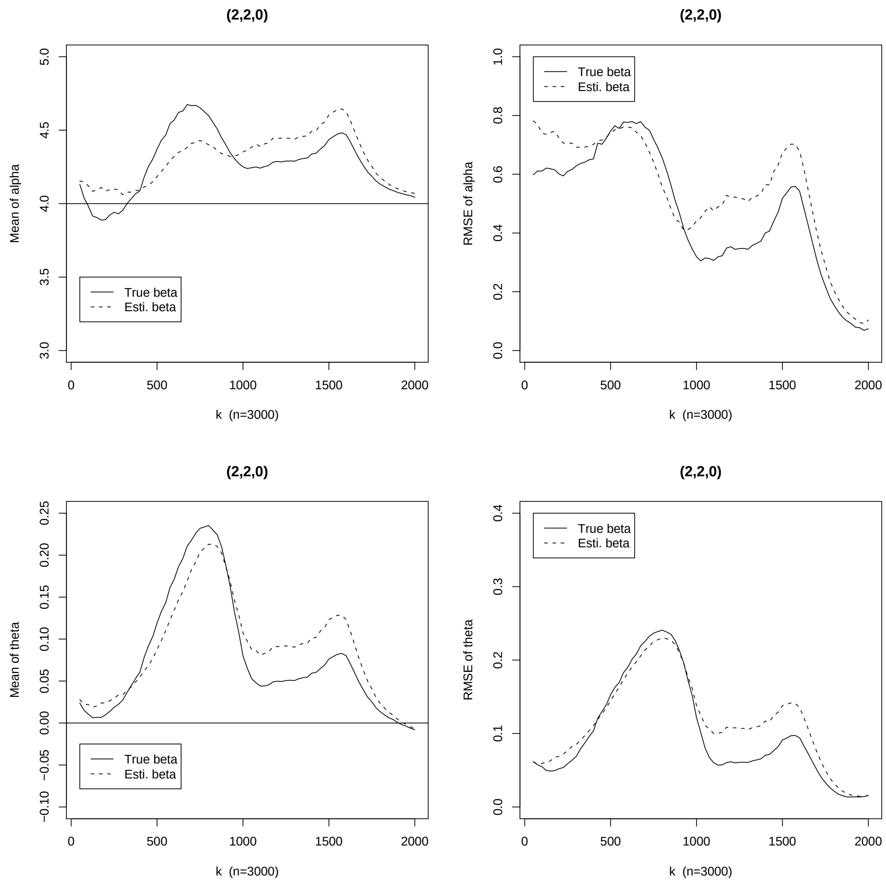

Figure 2: Mean and root of mean squared error (RMSE) are plotted against different k for the case of $(\tau_1, \tau_2, \theta) = (2, 2, 0)$ and $n = 3000$.

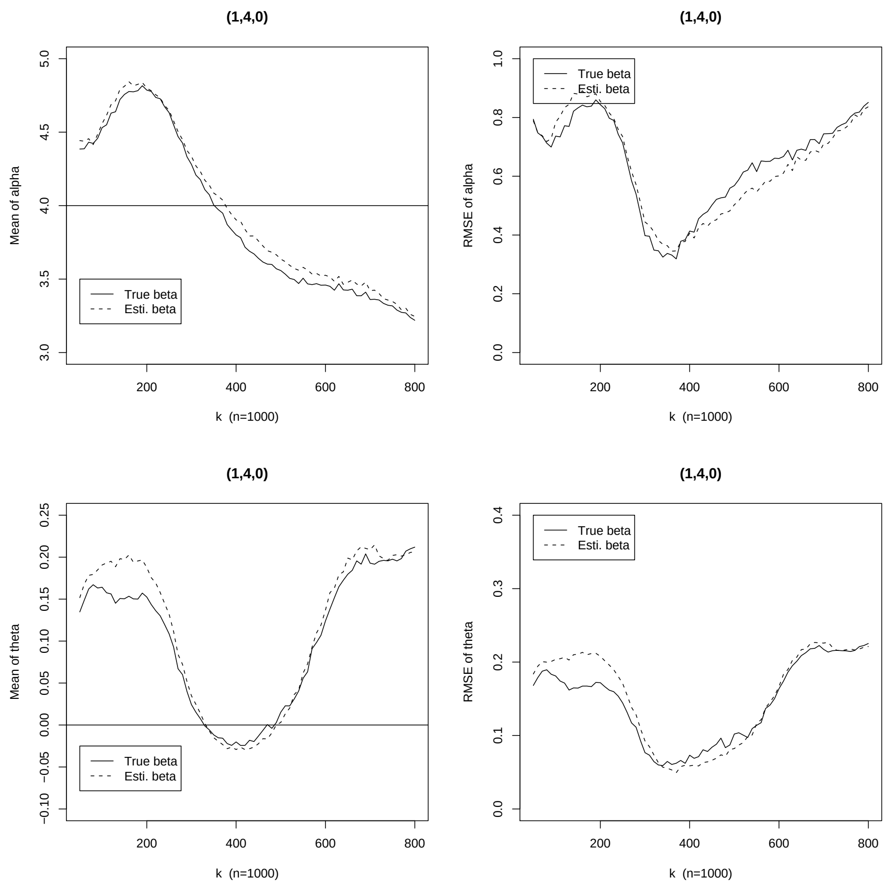

Figure 3: Mean and root of mean squared error (RMSE) are plotted against different k for the case of $(\tau_1, \tau_2, \theta) = (1, 4, 0)$ and $n = 1000$.

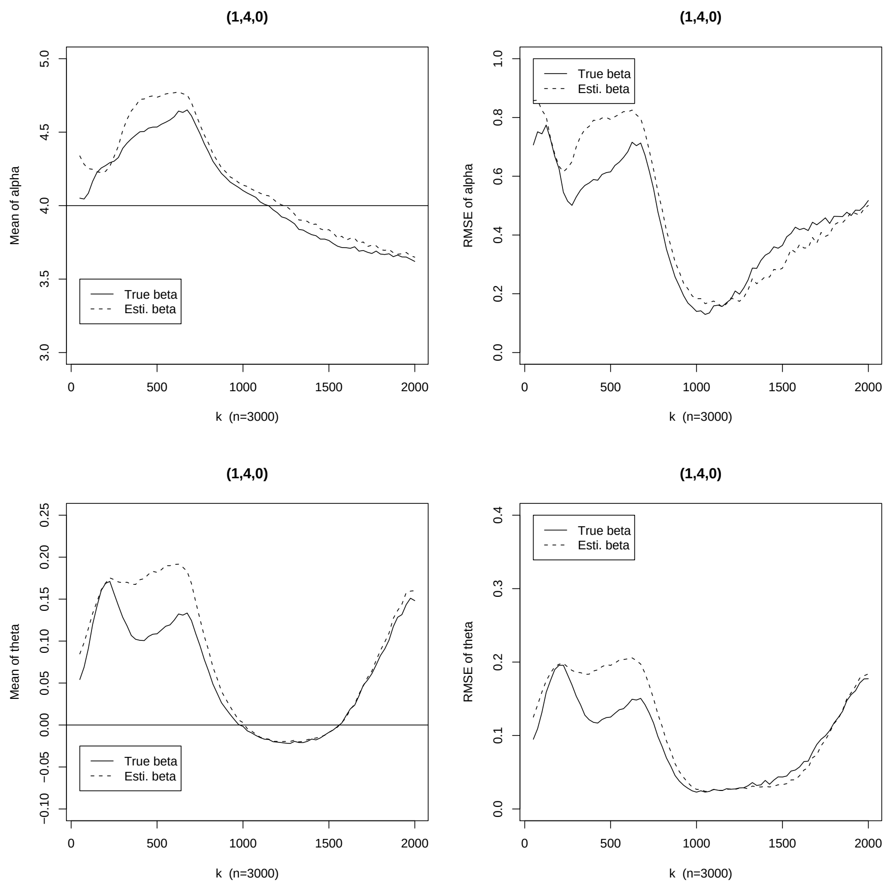

Figure 4: Mean and root of mean squared error (RMSE) are plotted against different k for the case of $(\tau_1, \tau_2, \theta) = (1, 4, 0)$ and $n = 3000$.

$$\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial d} = \sum_{i=0}^{k-1} \frac{\beta^*(\theta - Z_i)^{\beta^*-1}}{c\alpha(\theta - Z_i)^{\alpha-1} + d\beta^*(\theta - Z_i)^{\beta^*-1}} - \frac{(n-k)(\theta - Z_k)^{\beta^*}}{1 - c(\theta - Z_k)^{\alpha} - d(\theta - Z_k)^{\beta^*}} = 0, \tag{4.2}$$

$$\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \alpha} = \sum_{i=0}^{k-1} \frac{c(\theta - Z_i)^{\alpha-1}\{1 + \alpha\log(\theta - Z_i)\}}{c\alpha(\theta - Z_i)^{\alpha-1} + d\beta^*(\theta - Z_i)^{\beta^*-1}} - \frac{(n-k)c(\theta - Z_k)^{\alpha}\log(\theta - Z_k)}{1 - c(\theta - Z_k)^{\alpha} - d(\theta - Z_k)^{\beta^*}} = 0, \tag{4.3}$$

$$\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \beta^*} = \sum_{i=0}^{k-1} \frac{d(\theta - Z_i)^{\beta^*-1}\{1 + \beta^*\log(\theta - Z_i)\}}{c\alpha(\theta - Z_i)^{\alpha-1} + d\beta^*(\theta - Z_i)^{\beta^*-1}} - \frac{(n-k)d(\theta - Z_k)^{\beta^*}\log(\theta - Z_k)}{1 - c(\theta - Z_k)^{\alpha} - d(\theta - Z_k)^{\beta^*}} = 0 \tag{4.4}$$

and

$$\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \theta} = \sum_{i=0}^{k-1} \frac{c\alpha(\alpha-1)(\theta - Z_i)^{\alpha-2} + d\beta^*(\beta^*-1)(\theta - Z_i)^{\beta^*-2}}{c\alpha(\theta - Z_i)^{\alpha-1} + d\beta^*(\theta - Z_i)^{\beta^*-1}} - \frac{(n-k)\{c\alpha(\theta - Z_k)^{\alpha-1} + d\beta^*(\theta - Z_k)^{\beta^*-1}\}}{1 - c(\theta - Z_k)^{\alpha} - d(\theta - Z_k)^{\beta^*}} = 0. \tag{4.5}$$

It is easy to verify that

$$\begin{cases} \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial c} = 0 \\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial d} = 0 \\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \alpha} = 0 \\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \beta^*} = 0 \\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \theta} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial c} = 0 \\ c\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial c} + d\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial d} = 0 \\ \alpha\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \alpha} + \beta^*\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \beta^*} = 0 \\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \alpha} + \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \beta^*} = 0 \\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial \theta} = 0 \end{cases}$$

From $c\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial c} + d\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial d} = 0$, we have

$$c(\theta - Z_k)^{\alpha} + d(\theta - Z_k)^{\beta^*} = k/n. \tag{4.6}$$

Using $\alpha\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial\alpha} - c\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial c} = 0$, we have

$$\begin{aligned}&\sum_{i=0}^{k-1}\frac{c\alpha(\theta-Z_i)^{\alpha-1}\log(\theta-Z_i)}{c\alpha(\theta-Z_i)^{\alpha-1}+d\beta^*(\theta-Z_i)^{\beta^*-1}}\\ &=\frac{(n-k)c(\theta-Z_k)^{\alpha}\{\log(\theta-Z_k)-1/\alpha\}}{1-c(\theta-Z_k)^{\alpha}-d(\theta-Z_k)^{\beta^*}}.\end{aligned} \tag{4.7}$$

By $\beta^*\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial\beta^*} - d\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial d} = 0$, we have

$$\begin{aligned}&\sum_{i=0}^{k-1}\frac{d\beta(\theta-Z_i)^{\beta^*-1}\log(\theta-Z_i)}{c\alpha(\theta-Z_i)^{\alpha-1}+d\beta^*(\theta-Z_i)^{\beta^*-1}}\\ &=\frac{(n-k)d(\theta-Z_k)^{\beta^*}\{\log(\theta-Z_k)-1/\beta^*\}}{1-c(\theta-Z_k)^{\alpha}-d(\theta-Z_k)^{\beta^*}}.\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) and using (4.6), we have

$$\sum_{i=0}^{k-1}\log(\theta-Z_i)=k\log(\theta-Z_k)-n\{\alpha^{-1}c(\theta-Z_k)^{\alpha}+(\beta^*)^{-1}d(\theta-Z_k)^{\beta^*}\}. \tag{4.9}$$

It follows from (4.6) and (4.9) that

$$\begin{cases}c(\theta-Z_k)^{\alpha} &= \frac{k}{n}\frac{\beta^*}{\beta^*-\alpha}\{\frac{\alpha}{k}\sum_{i=0}^{k-1}\log(\frac{\theta-Z_k}{\theta-Z_i})-\frac{\alpha}{\beta^*}\}\\ d(\theta-Z_k)^{\beta^*} &= \frac{k}{n}\frac{\beta^*}{\beta^*-\alpha}\{1-\frac{\alpha}{k}\sum_{i=0}^{k-1}\log(\frac{\theta-Z_k}{\theta-Z_i})\},\end{cases} \tag{4.10}$$

i.e.

$$\begin{cases}c &= \frac{k}{n}\frac{\beta^*}{\beta^*-\alpha}\{\alpha T_n-\frac{\alpha}{\beta^*}\}(\theta-Z_k)^{-\alpha}\\ d &= \frac{k}{n}\frac{\beta^*}{\beta^*-\alpha}\{1-\alpha T_n\}(\theta-Z_k)^{-\beta^*}.\end{cases} \tag{4.11}$$

By (4.11), it is straightforward to check that

$$\begin{aligned}\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial\alpha}+\frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial\beta^*}&=0\Leftrightarrow\int_0^1 Q_{1n}(t;\theta,\alpha,\beta^*)dt=0\\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial\theta}&=0\Leftrightarrow\int_0^1 Q_{2n}(t;\theta,\alpha,\beta^*)dt=0\\ \frac{\partial \log L_1(\theta,\alpha,\beta^*,c,d)}{\partial c}&=0\Leftrightarrow\int_0^1 Q_{3n}(t;\theta,\alpha,\beta^*)dt=0.\end{aligned}$$

Hence the new bias reduction estimator $(\hat\theta(k),\hat\alpha(k),\hat\beta^*(k))$ is the solution to the equations (2.1).

For simplicity, throughout we denote $\hat{\alpha}=\hat{\alpha}(k), \hat{\beta}^*=\hat{\beta}^*(k), \hat{\theta}=\hat{\theta}(k), \hat{Q}_{jn}(t)=Q_{jn}(t;\hat{\theta}(k),\hat{\alpha}(k),\hat{\beta}^*(k))$ and $Q_{jn}(t)=Q_{jn}(t;\theta_0,\alpha_0,\beta_0^*)$ for $j=1,2,3$. Further we assume the consistency of $(\hat{\alpha},\hat{\beta}^*,\hat{\theta})$, i.e. $\hat{\alpha}-\alpha_0\xrightarrow{p}0$, $\hat{\beta}^*-\beta_0^*\xrightarrow{p}0$ and $\frac{\hat{\theta}-\theta_0}{a(n/k)}\xrightarrow{p}0$. Then by delta method, we have

$$
\begin{cases}
0=\dfrac{k^{1/2}}{1-\alpha T_n}\displaystyle\int_0^1\hat{Q}_{1n}(t)dt\\
\quad=\dfrac{k^{1/2}}{1-\alpha T_n}\displaystyle\int_0^1 Q_{1n}(t)dt+\dfrac{k^{1/2}(\hat{\alpha}-\alpha_0)}{1-\alpha T_n}\displaystyle\int_0^1\dfrac{\partial Q_{1n}(t)}{\partial\alpha}dt\\
\qquad+\dfrac{k^{1/2}A(n/k)(\hat{\beta}^*-\beta_0^*)}{1-\alpha T_n}\displaystyle\int_0^1\dfrac{1}{A(n/k)}\dfrac{\partial Q_{1n}(t)}{\partial\beta^*}dt\\
\qquad+\dfrac{k^{1/2}(\hat{\theta}-\theta_0)}{a(n/k)}\displaystyle\int_0^1\dfrac{a(n/k)}{1-\alpha T_n}\dfrac{\partial Q_{1n}(t)}{\partial\theta}dt+o_p(1),\\
0=k^{1/2}\displaystyle\int_0^1\hat{Q}_{2n}(t)dt\\
\quad=k^{1/2}\displaystyle\int_0^1 Q_{2n}(t)dt+k^{1/2}(\hat{\alpha}-\alpha_0)\displaystyle\int_0^1\dfrac{\partial Q_{2n}(t)}{\partial\alpha}dt\\
\qquad+k^{1/2}A(n/k)(\hat{\beta}^*-\beta_0^*)\displaystyle\int_0^1\dfrac{1}{A(n/k)}\dfrac{\partial Q_{2n}(t)}{\partial\beta^*}dt\\
\qquad+\dfrac{k^{1/2}(\hat{\theta}-\theta_0)}{a(n/k)}\displaystyle\int_0^1 a(n/k)\dfrac{\partial Q_{2n}(t)}{\partial\theta}dt+o_p(1),\\
0=\dfrac{k^{1/2}}{1-\alpha T_n}\displaystyle\int_0^1\hat{Q}_{3n}(t)dt\\
\quad=\dfrac{k^{1/2}}{1-\alpha T_n}\displaystyle\int_0^1 Q_{3n}(t)dt+\dfrac{k^{1/2}(\hat{\alpha}-\alpha_0)}{1-\alpha T_n}\displaystyle\int_0^1\dfrac{\partial Q_{3n}(t)}{\partial\alpha}dt\\
\qquad+\dfrac{k^{1/2}A(n/k)(\hat{\beta}^*-\beta_0^*)}{1-\alpha T_n}\displaystyle\int_0^1\dfrac{1}{A(n/k)}\dfrac{\partial Q_{3n}(t)}{\partial\beta_0^*}dt\\
\qquad+\dfrac{k^{1/2}(\hat{\theta}-\theta_0)}{a(n/k)}\displaystyle\int_0^1\dfrac{a(n/k)}{1-\alpha T_n}\dfrac{\partial Q_{3n}(t)}{\partial\theta}dt+o_p(1).
\end{cases}
\tag{4.12}
$$

The following steps are used to derive the asymptotic limit: i) approximate $\frac{\partial Q_{jn}(t)}{\partial\alpha}$, $\frac{\partial Q_{jn}(t)}{\partial\beta^*}$ and $\frac{\partial Q_{jn}(t)}{\partial\theta}$ for $j=1,2,3$; (ii) approximate $k^{1/2}\int_0^1 Q_{jn}(t)dt$ for $j=1,2,3$; (iii) approximate $k^{1/2}(\hat{\alpha}-\alpha_0)$, $k^{1/2}A(n/k)(\hat{\beta}^*-\beta_0^*)$ and $k^{1/2}\frac{\hat{\theta}-\theta_0}{a(n/k)}$ by the results in step (i) and (ii).

Before proving our theorem, we need some lemmas. The first three lemmas come from Li and Peng (2009a). Recall $Z_{[kt]}=X_{n,n-[kt]}$, and define $\Theta_n(t)=$

$(\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^* - \alpha}$. Then we write

$$\begin{aligned} Q_{1n}(t) &= \frac{(\alpha T_n - \frac{\alpha}{\beta^*}) + (1 - \alpha T_n)\Theta_n(t)}{\alpha(\alpha T_n - \frac{\alpha}{\beta^*}) + \beta_0^*(1 - \alpha T_n)\Theta_n(t)} + \log\Big(\frac{\theta - Z_{[kt]}}{\theta - Z_k}\Big) \\ &=: u_{1n}(t)/u_{2n}(t) + u_{3n}(t), \end{aligned} \tag{4.13}$$

$$\begin{aligned} Q_{2n}(t) &= \frac{\alpha(\alpha - 1)(\alpha T_n - \frac{\alpha}{\beta^*}) + \beta^*(\beta^* - 1)(1 - \alpha T_n)\Theta_n(t)}{(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\{\alpha(\alpha T_n - \frac{\alpha}{\beta^*}) + \beta^*(1 - \alpha T_n)\Theta_n(t)\}} \\ &\quad - \{\alpha + \beta^* - \alpha\beta^* T_n\} \\ &=: v_{1n}(t)/v_{2n}(t) - v_{3n}(t) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} Q_{3n}(t) &= \frac{\alpha(\beta^* - \alpha)}{\alpha\beta^*(\alpha T_n - \frac{\alpha}{\beta^*}) + (\beta^*)^2(1 - \alpha T_n)\Theta_n(t)} - 1 \\ &=: w_{1n}(t)/w_{2n}(t) - w_{3n}(t). \end{aligned} \tag{4.15}$$

Lemma 4.1. *Under the conditions of Theorem 2.2, we have*

$$\frac{\theta - X_{n,n-[kt]}}{\theta - X_{n,n-k}} = t^{-\gamma}\{1 + \tilde{\Delta}_n(t)\}$$

uniformly for $t \in [0, 1]$, *where*

$$\begin{aligned} \tilde{\Delta}_n(t) &= -\gamma t^{-1} k^{-1/2}\{W(t) - tW(1)\} + \tilde{A}(n/k)\frac{\gamma(t^{-\rho} - 1)}{\rho(\gamma + \rho)} \\ &\quad + \gamma\tilde{A}(n/k)\tilde{B}(n/k)\{t^{\gamma}[R_{\gamma,\rho,\eta}(1/t) - R_{\gamma,\rho,\eta}(\infty)] + R_{\gamma,\rho,\eta}(\infty)\} \\ &\quad + t^{-1/2-\varepsilon} o_p(k^{-1/2} + |A(n/k)B(n/k)|). \end{aligned}$$

Proof. See the proof of Lemma 4.2 of Li and Peng (2008).

Lemma 4.2. *Under the conditions of Theorem 2.2,*

$$\frac{\int_0^1 \Theta_n(t)dt - \frac{\alpha}{\beta^*}}{1 - \alpha T_n} \xrightarrow{p} \frac{\beta^* - \alpha}{2\beta^* - \alpha}$$

and

$$\frac{1 - \alpha T_n}{\epsilon_n} \xrightarrow{p} -\frac{\alpha(\beta^* - \alpha)}{\beta^*},$$

where

$$\epsilon_n = \frac{\gamma}{\rho(\gamma + \rho)} A(n/k) = \frac{-\alpha}{(\beta^* - \alpha)(\beta^* - \alpha + 1)} A(n/k).$$

Proof. It follows from Lemma 4.3 and its proof of Li and Peng (2008).

Lemma 4.3. *Under the conditions of Theorem 2.2,*

$$\begin{aligned}&\int_0^1 \Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\,dt\\ &= -\frac{\alpha}{(\beta^*)^2} + \frac{\alpha(\beta^*-\alpha)(\alpha^2+(\beta^*)^2-3\alpha\beta^*)}{(2\beta^*-\alpha)^2(\beta^*)^2}\epsilon_n + o_p(A(n/k)),\end{aligned}$$

$$\int_0^1 (\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{-1}\Theta_n(t)\,dt \xrightarrow{p} \frac{\alpha}{\beta^*-1},$$

$$\int_0^1 (\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{-1}\Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\,dt \xrightarrow{p} -\frac{\alpha}{(\beta^*-1)^2}$$

and

$$\int_0^1 \{\alpha\beta^* - (\beta^*)^2\Theta_n(t)\}^2\,dt = \frac{\alpha(\beta^*)^2(\beta^*-\alpha)^2}{2\beta^*-\alpha} + O_p(\epsilon_n).$$

Proof. Put $\tau_n(t) = \epsilon_n(t^{-\rho}-1)$. Then it follows from Lemma 4.1, $\beta^* = \beta - \alpha$, $\gamma = -1/\alpha$ and $\rho = -\beta/\alpha$ that

$$\begin{aligned}&\int_0^1 \Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\,dt\\ &= \int_0^1 t^{\beta/\alpha}\{1+\beta\tau_n(t)+o_p(A(n/k))\}\{-\gamma\log t+\tau_n(t)+o_p(A(n/k))\}\,dt\\ &= \int_0^1 t^{\beta/\alpha}\{-\gamma\log t + t^{-\rho}\epsilon_n - \epsilon_n - \gamma\beta\epsilon_n t^{-\rho}\log t + \gamma\beta\epsilon_n\log t\}dt + o_p(A(n/k))\\ &= \frac{\gamma}{(\beta/\alpha+1)^2} + \{\frac{1}{\beta/\alpha-\rho+1} - \frac{1}{\beta/\alpha+1} + \frac{\gamma\beta}{(\beta/\alpha-\rho+1)^2} - \frac{\gamma\beta}{(\beta/\alpha+1)^2}\}\epsilon_n\\ &\quad + o_p(A(n/k))\\ &= -\frac{\alpha}{(\beta^*)^2} + \frac{\alpha(\beta^*-\alpha)(\alpha^2+(\beta^*)^2-3\alpha\beta^*)}{(2\beta^*-\alpha)^2(\beta^*)^2}\epsilon_n + o_p(A(n/k)).\end{aligned}$$

The rest can be shown in a similar way.

Lemma 4.4. *Under the conditions of Theorem 2.2,*

$$\int_0^1 \frac{1}{(1-\alpha T_n)^{I_{\{j\neq 2\}}}}\frac{\partial Q_{jn}(t)}{\partial\alpha}dt \xrightarrow{p} \sigma_{j1}$$

$$\int_0^1 \frac{1}{(1-\alpha T_n)^{I_{\{j\neq 2\}}}}\frac{1}{A(n/k)}\frac{\partial Q_{jn}(t)}{\partial\beta_0^*}dt \xrightarrow{p} \sigma_{j2}$$

$$\int_0^1 \frac{a(n/k)}{(1-\alpha T_n)^{I_{\{j\neq 2\}}}}\frac{\partial Q_{jn}(t)}{\partial\theta}dt \xrightarrow{p} \sigma_{j3}$$

for $j = 1, 2, 3$, where σ_{ji} is defined in Theorem 2.2.

Proof. The proof follows from the following three steps. The first step is to approximate $\int_0^1 \frac{\partial Q_{1n}(t)}{\partial \cdot} dt$.

Recall that $u_{in}(t)$ for $i = 1, 2, 3$ are defined as in (4.13). Since $T_n \xrightarrow{p} \alpha^{-1}$, it is easy to check that $u_{1n}(t) \xrightarrow{p} 1 - \alpha/\beta^*$ and $u_{2n}(t) \xrightarrow{p} \alpha(1 - \alpha/\beta^*)$ uniformly for $t \in [0, 1]$. Straightforward calculations show that

$$
\begin{cases}
\dfrac{\partial u_{1n}(t)}{\partial \alpha} = -1/\beta^* + T_n(1 - \Theta_n(t)) - (1 - \alpha T_n)\Theta_n(t) \log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k}), \\
\dfrac{\partial u_{2n}(t)}{\partial \alpha} = 2\alpha T_n - 2\alpha/\beta^* - \beta^* \Theta_n(t) \Big[T_n + (1 - \alpha T_n) \log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k}) \Big], \\
\dfrac{\partial u_{3n}(t)}{\partial \alpha} = 0, \\
\dfrac{\partial u_{1n}(t)}{\partial \beta^*} = \alpha/(\beta^*)^2 + (1 - \alpha T_n)\Theta_n(t) \log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k}), \\
\dfrac{\partial u_{2n}(t)}{\partial \beta^*} = \alpha^2/(\beta^*)^2 + (1 - \alpha T_n)\Theta_n(t) \Big[1 + \beta^* \log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k}) \Big], \\
\dfrac{\partial u_{3n}(t)}{\partial \beta^*} = 0, \\
\dfrac{\partial u_{1n}(t)}{\partial \theta} = \dfrac{\alpha^2 t_\theta}{\theta - Z_k} [1 - \Theta_n(t)] \\
\qquad + (\beta^* - \alpha)(1 - \alpha T_n)(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^* - \alpha - 1} \dfrac{Z_{[kt]} - Z_k}{(\theta - Z_k)^2}, \\
\dfrac{\partial u_{2n}(t)}{\partial \theta} = \dfrac{\alpha^2 t_\theta}{\theta - Z_k} \big[\alpha - \beta^* \Theta_n(t) \big] \\
\qquad + \beta^*(1 - \alpha T_n)(\beta^* - \alpha)(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^* - \alpha - 1} \dfrac{Z_{[kt]} - Z_k}{(\theta - Z_k)^2}, \\
\dfrac{\partial u_{3n}(t)}{\partial \theta} = \dfrac{Z_{[kt]} - Z_k}{(\theta - Z_{[kt]})(\theta - Z_k)},
\end{cases}
$$

where

$$
t_\theta = \frac{\theta - Z_k}{\alpha} \frac{\partial T_n}{\partial \theta} = \alpha^{-1} \int_0^1 \Big\{ 1 - \frac{\theta - Z_k}{\theta - Z_{[kt]}} \Big\} dt \xrightarrow{p} \frac{-1}{\alpha(\alpha - 1)}.
$$

First, let's consider $\frac{1}{1-\alpha T_n}\int_0^1 \frac{\partial Q_{1n}(t)}{\partial \alpha}dt$. Write

$$\begin{aligned}
u_{1n}(t) &= (1-\frac{\alpha}{\beta^*}) - (1-\alpha T_n) + (1-\alpha T_n)\Theta_n(t), \\
u_{2n}(t) &= \alpha(1-\frac{\alpha}{\beta^*}) - \alpha(1-\alpha T_n) + \beta^*(1-\alpha T_n)\Theta_n(t), \\
\frac{\partial u_{1n}(t)}{\partial \alpha} &= (\frac{1}{\alpha} - \frac{1}{\beta^*} - \frac{1}{\alpha}\Theta_n(t)) - \alpha^{-1}(1-\alpha T_n)(1-\Theta_n(t)) \\
&\quad - (1-\alpha T_n)\Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k}), \\
\frac{\partial u_{2n}(t)}{\partial \alpha} &= 2(1-\frac{\alpha}{\beta^*}) - \frac{\beta^*}{\alpha}\Theta_n(t) - 2(1-\alpha T_n) + \frac{\beta^*}{\alpha}(1-\alpha T_n)\Theta_n(t) \\
&\quad - \beta^*(1-\alpha T_n)\Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k}).
\end{aligned}$$

We find that the constant item (i.e. item without $(1-\alpha T_n)$) in $\frac{\partial u_{1n}(t)}{\partial \alpha}u_{2n}(t) - u_{1n}(t)\frac{\partial u_{2n}(t)}{\partial \alpha}$ is

$$\begin{aligned}
&\{\frac{1}{\alpha} - \frac{1}{\beta^*} - \frac{1}{\alpha}\Theta_n(t)\}\{\alpha(1-\frac{\alpha}{\beta^*})\} - \{1-\frac{\alpha}{\beta^*}\}\{2(1-\frac{\alpha}{\beta^*}) - \frac{\beta^*}{\alpha}\Theta_n(t)\} \\
=&\frac{(\beta^*-\alpha)^2}{\alpha\beta^*}\{\Theta_n(t) - \frac{\alpha}{\beta^*}\} =: W,
\end{aligned}$$

the item involving $(1-\alpha T_n)$ in $\frac{\partial u_{1n}(t)}{\partial \alpha}u_{2n}(t) - u_{1n}(t)\frac{\partial u_{2n}(t)}{\partial \alpha}$ is

$$\begin{aligned}
&(1-\alpha T_n)\Big\{\{\{\frac{1}{\alpha} - \frac{1}{\beta^*} - \frac{1}{\alpha}\Theta_n(t)\}\{\beta^*\Theta_n(t) - \alpha\} \\
&+ \alpha(1-\frac{\alpha}{\beta^*})\{-\alpha^{-1}(1-\Theta_n(t)) - \Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\} \\
&- \{1-\frac{\alpha}{\beta^*}\}\{-2 + \frac{\beta^*}{\alpha}\Theta_n(t) - \beta^*\Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\} \\
&- \{2(1-\frac{\alpha}{\beta^*}) - \frac{\beta^*}{\alpha}\Theta_n(t)\}\{\Theta_n(t) - 1\}\Big\} \\
=&(1-\alpha T_n)\{2(1-\frac{\alpha}{\beta^*}) + (\frac{\alpha}{\beta^*} - \frac{\beta^*}{\alpha})\Theta_n(t) + \frac{(\beta^*-\alpha)^2}{\beta^*}\Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\} \\
=:&S,
\end{aligned}$$

and the other terms in $\frac{\partial u_{1n}(t)}{\partial \alpha}u_{2n}(t) - u_{1n}(t)\frac{\partial u_{2n}(t)}{\partial \alpha}$ is $o_p(1-\alpha T_n)$. It follows from the above approximations, Lemmas 4.2 and 4.3 that

$$
\begin{aligned}
&\int_0^1 \frac{1}{1-\alpha T_n}\frac{\partial Q_{1n}(t)}{\partial \alpha}dt \\
&= \int_0^1 \frac{1}{(1-\alpha T_n)u_{2n}^2(t)}\Big\{\frac{\partial u_{1n}(t)}{\partial \alpha}u_{2n}(t) - u_{1n}(t)\frac{\partial u_{2n}(t)}{\partial \alpha}\Big\}dt \\
&= \int_0^1 \frac{1}{\alpha^2(1-\alpha/\beta^*)^2}\{\frac{W}{1-\alpha T_n} + \frac{S}{1-\alpha T_n}\}dt + o_p(1) \\
&\xrightarrow{p} \frac{(\beta^*)^2}{\alpha^2(\beta^*-\alpha)^2}\Big\{\frac{(\beta^*-\alpha)^2}{\alpha\beta^*}\frac{\beta^*-\alpha}{2\beta^*-\alpha} + 2(1-\frac{\alpha}{\beta^*}) + (\frac{\alpha}{\beta^*} - \frac{\beta^*}{\alpha})\frac{\alpha}{\beta^*} \\
&\qquad + \frac{(\beta^*-\alpha)^2}{\beta^*}(-\frac{\alpha}{(\beta^*)^2})\Big\} \\
&= \frac{(\beta^*)^2}{\alpha^2(\beta^*-\alpha)^2}\Big\{\frac{(\beta^*-\alpha)^3}{\alpha\beta^*(2\beta^*-\alpha)} + \frac{(\beta^*-\alpha)^3}{(\beta^*)^3}\Big\} \\
&= \sigma_{11}.
\end{aligned}
$$

Secondly, let's consider $\frac{1}{1-\alpha T_n}\frac{1}{A(n/k)}\int_0^1 \frac{\partial Q_{1n}(t)}{\partial \beta^*}dt$. We find that the constant item (i.e. without $(1-\alpha T_n)$) in $\frac{\partial u_{1n}(t)}{\partial \beta^*}u_{2n}(t) - u_{1n}(t)\frac{\partial u_{2n}(t)}{\partial \beta^*}$ is zero, the item involving $(1-\alpha T_n)$ is

$$
\begin{aligned}
&(1-\alpha T_n)\Big\{\frac{\alpha}{(\beta^*)^2}(\beta^*\Theta_n(t)-\alpha) + \alpha(1-\frac{\alpha}{\beta^*})\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k}) \\
&\quad - (1-\frac{\alpha}{\beta^*})\Theta_n(t)\Big(1+\beta^*\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\Big) - \frac{\alpha^2}{(\beta^*)^2}(\Theta_n(t)-1)\Big\} \\
=&(1-\alpha T_n)\Big\{-\frac{(\beta^*-\alpha)^2}{(\beta^*)^2}\Theta_n(t) - \frac{(\beta^*-\alpha)^2}{\beta^*}\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\Big\},
\end{aligned}
$$

and the item involving $(1-\alpha T_n)^2$ is

$$
\begin{aligned}
&(1-\alpha T_n)^2\Big\{\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\big[\beta^*\Theta_n(t)-\alpha\big] \\
&\qquad - (\Theta_n(t)-1)\Theta_n(t)\big[1+\beta^*\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\big]\Big\} \\
=&(1-\alpha T_n)^2\Big\{(\beta^*-\alpha)\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k}) - \Theta_n(t)^2 + \Theta_n(t)\Big\}.
\end{aligned}
$$

Thus, by Lemmas 4.2 and 4.3,

$$
\begin{aligned}
&\int_0^1 \frac{1}{1-\alpha T_n}\frac{\partial Q_{1n}(t)}{\partial\beta^*}dt\\
=&\frac{(\beta^*)^2(1+o_p(1))}{\alpha^2(\beta^*-\alpha)^2}\int_0^1\Big\{-\frac{(\beta^*-\alpha)^2}{(\beta^*)^2}\Theta_n(t)-\frac{(\beta^*-\alpha)^2}{\beta^*}\Theta_n(t)\log\Big(\frac{\theta-Z_{[kt]}}{\theta-Z_k}\Big)\\
&+(1-\alpha T_n)\big\{(\beta^*-\alpha)\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})-\Theta_n(t)^2+\Theta_n(t)\big\}\Big\}dt\\
=&\frac{(\beta^*)^2(1+o_p(1))}{\alpha^2(\beta^*-\alpha)^2}\Big\{-\frac{(\beta^*-\alpha)^2}{(\beta^*)^2}\big(\frac{\alpha}{\beta^*}+(1-\alpha T_n)\frac{\beta^*-\alpha}{2\beta^*-\alpha}\big)\\
&-\frac{(\beta^*-\alpha)^2}{\beta^*}\big(-\frac{\alpha}{(\beta^*)^2}+\epsilon_n\frac{\alpha(\beta^*-\alpha)(\alpha^2+(\beta^*)^2-3\alpha\beta^*)}{(2\beta^*-\alpha)^2(\beta^*)^2}\big)\\
&+(1-\alpha T_n)\big\{\frac{\alpha(\alpha-\beta^*)}{(\beta^*)^2}-\frac{\alpha}{2\beta^*-\alpha}+\frac{\alpha}{\beta^*}+o_p(1)\big\}\Big\}\\
=&\frac{(\beta^*-\alpha)^2}{\alpha\beta^*(2\beta^*-\alpha)}\epsilon_n-\frac{(\beta^*-\alpha)(\alpha^2-3\alpha\beta^*+(\beta^*)^2)}{\alpha\beta^*(2\beta^*-\alpha)^2}\epsilon_n+\frac{\beta^*-\alpha}{\beta^*(2\beta^*-\alpha)}\epsilon_n\\
&\quad+o_p(A(n/k))\\
=&\frac{(\beta^*-\alpha)((\beta^*)^2+2\alpha\beta^*-\alpha^2)}{\alpha\beta^*(2\beta^*-\alpha)^2}\epsilon_n+o_p(A(n/k))\\
=&\frac{(\beta^*)^2+2\alpha\beta^*-\alpha^2}{\beta^*(2\beta-\alpha)^2(\alpha-\beta^*-1)}A(n/k)+o_p(A(n/k))\\
=&\sigma_{12}A(n/k)+o_p(A(n/k)).
\end{aligned}
$$

Thirdly, the limit of $\frac{a(n/k)}{1-\alpha T_n}\int_0^1\frac{\partial Q_{1n}(t)}{\partial\theta}dt$ follows from Lemma 4.4 of Li and Peng (2009a). Hence we finish the approximations in step one.

The second step in the proof is to approximate $\int_0^1\frac{\partial Q_{2n}(t)}{\partial\cdot}dt$ as follows.

Recall that $v_{in}(t)$ for $i=1,2,3$ are defined as in (4.14). Straightforward calculations lead to

$$
\begin{cases}
\dfrac{\partial v_{1n}(t)}{\partial \alpha} = (3\alpha-2)(\alpha T_n - \dfrac{\alpha}{\beta^*}) \\
\qquad\qquad - \beta^*(\beta^*-1)\Theta_n(t)\Big(T_n + (1-\alpha T_n)\log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k})\Big), \\
\dfrac{\partial v_{2n}(t)}{\partial \alpha} = \dfrac{\theta - Z_{[kt]}}{\theta - Z_k}\Big\{2\alpha T_n - 2\alpha/\beta^* \\
\qquad\qquad - \beta^*\Theta_n(t)\Big(T_n + (1-\alpha T_n)\log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k})\Big)\Big\}, \\
\dfrac{\partial v_{3n}(t)}{\partial \alpha} = 1 - \beta^* T_n, \\
\dfrac{\partial v_{1n}(t)}{\partial \beta^*} = \dfrac{\alpha^2(\alpha-1)}{(\beta^*)^2} \\
\qquad\qquad + (1-\alpha T_n)\Theta_n(t)\Big(2\beta^* - 1 + \beta^*(\beta^*-1)\log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k})\Big), \\
\dfrac{\partial v_{2n}(t)}{\partial \beta^*} = \dfrac{\theta - Z_{[kt]}}{\theta - Z_k}\Big\{\alpha^2/(\beta^*)^2 + (1-\alpha T_n)\Theta_n(t)\Big(1 + \beta^*\log(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k})\Big)\Big\}, \\
\dfrac{\partial v_{3n}(t)}{\partial \beta^*} = 1 - \alpha T_n, \\
\dfrac{\partial v_{1n}(t)}{\partial \theta} = \dfrac{\alpha^2 t_\theta}{\theta - Z_k}\big(\alpha(\alpha-1) - \beta^*(\beta^*-1)\Theta_n(t)\big) \\
\qquad\qquad + \beta^*(\beta^*-1)(\beta^*-\alpha)(1-\alpha T_n)(\dfrac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^*-\alpha-1}\dfrac{Z_{[kt]} - Z_k}{(\theta - Z_k)^2}, \\
\dfrac{\partial v_{2n}(t)}{\partial \theta} = \dfrac{Z_{[kt]} - Z_k}{(\theta - Z_k)^2} \cdot u_{2n}(t) + \dfrac{\theta - Z_{[kt]}}{\theta - Z_k} \cdot \dfrac{\partial u_{2n}(t)}{\partial \theta}, \\
\dfrac{\partial v_{3n}(t)}{\partial \theta} = -\dfrac{\alpha^2 \beta^* t_\theta}{\theta - Z_k}.
\end{cases}
$$

Write

$$
v_{1n}(t) = \alpha(\alpha-1)(1 - \frac{\alpha}{\beta^*}) - \alpha(\alpha-1)(1-\alpha T_n) + \beta^*(\beta^*-1)(1-\alpha T_n)\Theta_n(t),
$$

$$
v_{2n}(t) = \frac{\theta - Z_{[kt]}}{\theta - Z_k}\Big\{\alpha(1 - \frac{\alpha}{\beta^*}) - \alpha(1-\alpha T_n) + \beta^*(1-\alpha T_n)\Theta_n(t)\Big\}.
$$

First let's consider $\int_0^1 \frac{\partial Q_{2n}(t)}{\partial \alpha} dt$. Using the above expressions and Lemmas 4.2

and 4.3, we have

$$
\begin{aligned}
&\int_0^1 \frac{\partial Q_{2n}(t)}{\partial \alpha}\, dt \\
&= \int_0^1 \frac{\frac{\partial v_{1n}(t)}{\partial \alpha}\, v_{2n}(t) - v_{1n}(t)\,\frac{\partial v_{2n}(t)}{\partial \alpha}}{v_{2n}^2(t)}\, dt - \int_0^1 \frac{\partial v_{3n}(t)}{\partial \alpha}\, dt \\
=& \int_0^1 \Big(\frac{\theta - Z_k}{\theta - Z_{[kt]}}\Big)\frac{\beta^*}{\alpha(\beta^* - \alpha)}\Big\{(3\alpha - 2)(1 - \frac{\alpha}{\beta^*}) - \frac{\beta^*}{\alpha}(\beta^* - 1)\Theta_n(t) \\
&- (\alpha - 1)(2 - \frac{2\alpha}{\beta^*} - \frac{\beta^*}{\alpha}\Theta_n(t)) + o_p(1)\Big\}dt - (1 - \beta^* T_n) \\
\xrightarrow{p}& (3 - \frac{2}{\alpha})\int_0^1 t^{-1/\alpha}dt - \frac{(\beta^*)^2(\beta^* - 1)}{\alpha^2(\beta^* - \alpha)}\int_0^1 t^{-\frac{1}{\alpha}}t^{\frac{\beta^* - \alpha}{\alpha}}\, dt \\
&- \frac{\beta^*(\alpha - 1)}{\alpha(\beta^* - \alpha)}\int_0^1 t^{-1/\alpha}\Big\{2 - \frac{2\alpha}{\beta^*} - \frac{\beta^*}{\alpha}t^{\frac{\beta^* - \alpha}{\alpha}}\Big\}dt - (1 - \frac{\beta^*}{\alpha}) \\
=& \frac{3\alpha - 2}{\alpha - 1} - \frac{(\beta^*)^2}{\alpha(\beta^* - \alpha)} - \frac{\beta^*(\alpha - 1)}{\alpha(\beta^* - \alpha)}\{\frac{2\beta^* - 2\alpha}{\beta^*}\frac{\alpha}{\alpha - 1} - \frac{\beta^*}{\beta^* - 1}\} \\
&\quad - (1 - \frac{\beta^*}{\alpha}) \\
=& \sigma_{21}.
\end{aligned}
$$

Secondly, we consider $\frac{1}{A(n/k)}\int_0^1 \frac{\partial Q_{2n}(t)}{\partial \beta^*}dt$. As above, we obtain

$$\begin{aligned}
&\int_0^1 \frac{\partial Q_{2n}(t)}{\partial \beta^*}dt \\
=&\int_0^1 \Big(\frac{\theta - Z_{[kt]}}{\theta - Z_k}\Big)^{-1} \frac{1+o_p(1)}{\alpha^2(1-\alpha/\beta^*)^2}\Big\{\frac{\alpha^2(\alpha-1)}{(\beta^*)^2}\big[\alpha(1-\frac{\alpha}{\beta^*}) \\
&-\alpha(1-\alpha T_n)+\beta^*(1-\alpha T_n)\Theta_n(t)\big] \\
&+\alpha(1-\frac{\alpha}{\beta^*})\big[(1-\alpha T_n)\Theta_n(t)\{2\beta^*-1+\beta^*(\beta^*-1)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\}\big] \\
&-\frac{\alpha^2}{(\beta^*)^2}\big[\alpha(\alpha-1)(1-\frac{\alpha}{\beta^*})-\alpha(\alpha-1)(1-\alpha T_n) \\
&\qquad +\beta^*(\beta^*-1)(1-\alpha T_n)\Theta_n(t)\big] \\
&-\alpha(\alpha-1)(1-\frac{\alpha}{\beta^*})\big[(1-\alpha T_n)\Theta_n(t)\{1+\beta^*\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\}\big] \\
&+o_p(1-\alpha T_n)\Big\}dt-(1-\alpha T_n) \\
=&\int_0^1 (\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{-1}\frac{1+o_p(1)}{\alpha^2(1-\alpha/\beta^*)^2}\{(1-\alpha T_n)\Theta_n(t)\frac{2\alpha(\alpha-\beta^*)^2}{\beta^*} \\
&+(1-\alpha T_n)\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\alpha(\beta^*-\alpha)^2+o_p(1-\alpha T_n)\}\,dt \\
&\quad -(1-\alpha T_n).
\end{aligned}$$

By Lemmas 4.2 and 4.3, we have

$$\begin{aligned}
&\int_0^1 \frac{1}{A(n/k)}\frac{\partial Q_{2n}(t)}{\partial \beta^*}dt \\
=&-\frac{\beta^*}{(\beta^*-\alpha)^2(\alpha-\beta^*-1)}\Big(\frac{2\alpha(\alpha-\beta^*)^2}{\beta^*}\int_0^1(\frac{\theta-Z_{[kt]}}{\theta-Z_k})^{-1}\Theta_n(t)dt \\
&+\alpha(\beta^*-\alpha)^2\int_0^1(\frac{\theta-Z_{[kt]}}{\theta-Z_k})^{-1}\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})dt\Big)+\frac{\alpha^2}{\beta^*(\alpha-\beta^*-1)}+o_p(1) \\
=&-\frac{2\alpha}{\alpha-\beta^*-1}\cdot\frac{\alpha}{\beta^*-1}-\frac{\alpha\beta^*}{\alpha-\beta^*-1}\cdot\frac{-\alpha}{(\beta^*-1)^2}+\frac{\alpha^2}{\beta^*(\alpha-\beta^*-1)}+o_p(1) \\
=&\sigma_{22}+o_p(1).
\end{aligned}$$

Thirdly, the limit of $\int_0^1 a(n/k)\frac{\partial Q_{2n}(t)}{\partial \theta}dt$ follows from Li and Peng (2009a). Hence we finish the approximations in step two.

The third step is to approximate $\int_0^1 \frac{\partial Q_{3n}(t)}{\partial \cdot}dt$ as follows.

Recall that $w_{in}(t)$ for $i = 1, 2, 3$ are defined in (4.15). Since $T_n \xrightarrow{p} \alpha^{-1}$, we have $w_{2n}(t) \xrightarrow{p} \alpha(\beta^* - \alpha) = w_{1n}(t)$. It is straightforward to verify that

$$\begin{cases} \frac{\partial w_{1n}(t)}{\partial \alpha} & = \beta^* - 2\alpha, \\ \frac{\partial w_{2n}(t)}{\partial \alpha} & = 2\alpha\beta^* T_n - 2\alpha - (\beta^*)^2 \Theta_n(t)\Big(T_n + (1 - \alpha T_n)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\Big), \\ \frac{\partial w_{3n}(t)}{\partial \alpha} & = 0, \\ \frac{\partial w_{1n}(t)}{\partial \beta^*} & = \alpha, \\ \frac{\partial w_{2n}(t)}{\partial \beta^*} & = \alpha^2 T_n + \beta^*(1 - \alpha T_n)\Theta_n(t)\Big(2 + \beta^* \log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\Big), \\ \frac{\partial w_{3n}(t)}{\partial \beta^*} & = 0, \\ \frac{\partial w_{1n}(t)}{\partial \theta} & = 0, \\ \frac{\partial w_{2n}(t)}{\partial \theta} & = \frac{\alpha^2 \beta^* t_\theta}{\theta - Z_k}\big(\alpha - \beta^* \Theta_n(t)\big) \\ & \quad + (\beta^*)^2 (1 - \alpha T_n)(\beta^* - \alpha)(\frac{\theta - Z_{[kt]}}{\theta - Z_k})^{\beta^* - \alpha - 1} \frac{Z_{[kt]} - Z_k}{(\theta - Z_k)^2}, \\ \frac{\partial w_{3n}(t)}{\partial \theta} & = 0. \end{cases}$$

First, let's consider $\frac{1}{1-\alpha T_n}\int_0^1 \frac{\partial Q_{3n}(t)}{\partial \alpha} dt$. Since

$$\begin{aligned} & \frac{\partial w_{1n}(t)}{\partial \alpha} w_{2n}(t) - w_{1n}(t) \frac{\partial w_{2n}(t)}{\partial \alpha} \\ =& (\beta^* - 2\alpha)\{\alpha\beta^*(\alpha T_n - \frac{\alpha}{\beta^*}) + (\beta^*)^2 (1 - \alpha T_n)\Theta_n(t)\} \\ & - \alpha(\beta^* - \alpha)\big\{2\alpha\beta^* T_n - 2\alpha - (\beta^*)^2 \Theta_n(t)\big(T_n + (1 - \alpha T_n)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\big)\big\} \\ =& \alpha(\beta^*)^2 (1 - \alpha T_n) - \alpha(\beta^*)^2 (1 - \alpha T_n)\Theta_n(t) \\ & \quad + \alpha(\beta^*)^2 (\beta^* - \alpha)(1 - \alpha T_n)\Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k}) \\ & - (\alpha - \beta^*)(\beta^*)^2 (\Theta_n(t) - \alpha/\beta^*), \end{aligned}$$

it follows from Lemmas 4.2 and 4.3 that

$$\begin{aligned} & \int_0^1 \frac{1}{1 - \alpha T_n} \frac{\partial Q_{3n}(t)}{\partial \alpha} dt \\ =& \frac{(\beta^*)^2 (1 + o_p(1))}{\alpha(\beta^* - \alpha)^2} \int_0^1 \Big(1 - \Theta_n(t) + (\beta^* - \alpha)\Theta_n(t)\log(\frac{\theta - Z_{[kt]}}{\theta - Z_k})\Big) dt \\ & - \frac{(\beta^*)^2}{\alpha^2(\alpha - \beta^*)} \frac{\int_0^1 \Theta_n(t) dt - \alpha/\beta^*}{1 - \alpha T_n} + o_p(1) \\ \xrightarrow{p} & \frac{(\beta^*)^2}{\alpha(\beta^* - \alpha)^2}\big(1 - \frac{\alpha}{\beta^*} - \frac{\alpha(\beta^* - \alpha)}{(\beta^*)^2}\big) + \frac{(\beta^*)^2}{\alpha^2(2\beta^* - \alpha)} \\ =& \sigma_{31}. \end{aligned}$$

Secondly, we consider $\frac{1}{1-\alpha T_n}\frac{1}{A(n/k)}\int_0^1 \frac{\partial Q_{3n}(t)}{\partial \beta^*}dt$. Since

$$\begin{aligned}
&\frac{\partial w_{1n}(t)}{\partial \beta^*}w_{2n}(t)-w_{1n}(t)\frac{\partial w_{2n}(t)}{\partial \beta^*}\\
=&\alpha\{\alpha\beta^*(\alpha T_n-\frac{\alpha}{\beta^*})+(\beta^*)^2(1-\alpha T_n)\Theta_n(t)\}\\
&-\alpha(\beta^*-\alpha)\Big\{\alpha^2T_n+\beta^*(1-\alpha T_n)\Theta_n(t)\Big(2+\beta^*\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\Big)\Big\}\\
=&-\alpha^3(1-\alpha T_n)+\alpha\beta^*(2\alpha-\beta^*)(1-\alpha T_n)\Theta_n(t)\\
&-\alpha(\beta^*)^2(\beta^*-\alpha)(1-\alpha T_n)\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k}),
\end{aligned}$$

it follows Lemmas 4.2 and 4.3 that

$$\begin{aligned}
&\int_0^1\frac{1}{(1-\alpha T_n)A(n/k)}\frac{\partial Q_{3n}(t)}{\partial \beta^*}dt\\
=&\frac{1+o_p(1)}{\alpha^2(\beta^*-\alpha)^2A(n/k)}\int_0^1\Big\{-\alpha^3+\alpha\beta^*(2\alpha-\beta^*)\Theta_n(t)\\
&-\alpha(\beta^*)^2(\beta^*-\alpha)\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\Big\}dt\\
=&\frac{1+o_p(1)}{\alpha^2(\beta^*-\alpha)^2A(n/k)}\{\alpha\beta^*(2\alpha-\beta^*)(\int_0^1\Theta_n(t)\,dt-\frac{\alpha}{\beta^*})\\
&-\alpha(\beta^*)^2(\beta^*-\alpha)\big(\int_0^1\Theta_n(t)\log(\frac{\theta-Z_{[kt]}}{\theta-Z_k})\,dt+\frac{\alpha}{(\beta^*)^2}\big)\}\\
\xrightarrow{p}&\frac{1}{\alpha^2(\beta^*-\alpha)^2}\{\alpha\beta^*(2\alpha-\beta^*)\frac{\beta^*-\alpha}{2\beta^*-\alpha}(-\frac{\alpha(\beta^*-\alpha)}{\beta^*})\frac{-\alpha}{(\beta^*-\alpha)(\beta^*-\alpha+1)}\\
&-\alpha(\beta^*)^2(\beta^*-\alpha)\frac{\alpha(\beta^*-\alpha)(\alpha^2+(\beta^*)^2-3\alpha\beta^*)}{(2\beta^*-\alpha)^2(\beta^*)^2}\frac{-\alpha}{(\beta^*-\alpha)(\beta^*-\alpha+1)}\}\\
=&\sigma_{32}.
\end{aligned}$$

Thirdly, we consider $\frac{a(n/k)}{1-\alpha T_n}\int_0^1\frac{\partial Q_{3n}(t)}{\partial\theta}dt$. Since

$$\begin{aligned}
\frac{\partial Q_{3n}(t)}{\partial\theta}=&-\frac{w_{1n}(t)}{w_{2n}^2(t)}\frac{\partial w_{2n}(t)}{\partial\theta}\\
=&\frac{w_{1n}(t)}{w_{2n}^2(t)}(\theta-Z_k)^{-1}\Big\{\alpha^2(\beta^*)^2t_\theta\big(\Theta_n(t)-\frac{\alpha}{\beta^*}\big)\\
&+(\beta^*)^2(1-\alpha T_n)(\alpha-\beta^*)(\frac{\theta-Z_{[kt]}}{\theta-Z_k})^{\beta^*-\alpha-1}\\
&-(\beta^*)^2(1-\alpha T_n)(\alpha-\beta^*)(\frac{\theta-Z_{[kt]}}{\theta-Z_k})^{\beta^*-\alpha}\Big\},
\end{aligned}$$

$\frac{a(n/k)}{\theta - Z_k} \xrightarrow{p} \frac{1}{\alpha}$ and $t_\theta \xrightarrow{p} -\frac{1}{\alpha(\alpha-1)}$, it follows from Lemmas 4.1, 4.2 and 4.3 that

$$\begin{aligned}
&\int_0^1 \frac{a(n/k)}{1-\alpha T_n} \frac{\partial Q_{3n}(t)}{\partial \theta} dt \\
\xrightarrow{p} & \frac{\alpha(\beta^*-\alpha)}{\alpha^2(\beta^*-\alpha)^2} \frac{1}{\alpha} \Big\{ \alpha^2 (\beta^*)^2 \frac{-1}{\alpha(\alpha-1)} \frac{\beta^*-\alpha}{2\beta^*-\alpha} \\
& + (\beta^*)^2(\alpha-\beta^*) \int_0^1 \left[t^{(\beta^*-\alpha-1)/\alpha} - t^{(\beta^*-\alpha)/\alpha} \right] dt \Big\} \\
= & \sigma_{33}.
\end{aligned}$$

Hence, we finish the approximations in step three, i.e., we finish the proof of Lemma 4.4.

Lemma 4.5. *Under the conditions of Theorem 2.2, we have* $\sqrt{k}\frac{\int_0^1 Q_{1n}(t)dt}{1-\alpha T_n} + N_1 = o_P(1)$, $\sqrt{k}\int_0^1 Q_{2n}(t)dt + N_2 = o_P(1)$ *and* $\sqrt{k}\frac{\int_0^1 Q_{3n}(t)dt}{1-\alpha T_n} + N_3 = o_P(1)$, *where*

$$\begin{cases}
N_1 &= \frac{\beta^*(\beta^*-\alpha)}{\alpha^2} \int_0^1 \left\{ \frac{1}{\alpha} t^{\beta^*/\alpha - 2} - \frac{1}{2\beta^*-\alpha} t^{-1} \right\} \left\{ W(t) - tW(1) \right\} dt \\
N_2 &= \int_0^1 \left\{ \left(1-\frac{1}{\alpha}\right) t^{-1/\alpha-1} - \frac{\beta^*}{\beta^*-1} t^{-1} \right\} \left\{ W(t) - tW(1) \right\} dt \\
N_3 &= \frac{(\beta^*)^2}{\alpha^2} \int_0^1 t^{\beta^*/\alpha - 2} \left\{ W(t) - tW(1) \right\} dt.
\end{cases}$$

Proof. The formulas of N_1 and N_2 follows from Lemma 4.5 in Li and Peng (2009a). Write

$$\begin{aligned}
&\int_0^1 Q_{3n}(t)dt \\
=& \int_0^1 \frac{1}{1 - \frac{1}{\alpha(\beta^*-\alpha)}(1-\alpha T_n)(\alpha\beta^* - (\beta^*)^2 \Theta_n(t))} dt - 1 \\
=& \int_0^1 \Big\{ \frac{1-\alpha T_n}{\alpha(\beta^*-\alpha)} (\alpha\beta^* - (\beta^*)^2\Theta_n(t)) + \Big[\frac{1-\alpha T_n}{\alpha(\beta^*-\alpha)} (\alpha\beta^* - (\beta^*)^2 \Theta_n(t)) \Big]^2 \Big\} \\
& + o_p((1-\alpha T_n)^2).
\end{aligned}$$

Using Lemmas 4.1, 4.2 and 4.3, we have

$$\begin{aligned}
&\sqrt{k}\int_0^1 \frac{1}{1-\alpha T_n} Q_{3n}(t)dt \\
=&\sqrt{k}\Big\{\frac{\beta^*}{\beta^*-\alpha} - \frac{(\beta^*)^2}{\alpha(\beta^*-\alpha)}\big(\frac{\alpha}{\beta^*} - \frac{\alpha(\beta^*-\alpha)^2}{\beta^*(2\beta^*-\alpha)}\epsilon_n \\
&+ k^{-1/2}\frac{\beta^*-\alpha}{\alpha}\int_0^1 t^{\beta^*/\alpha-2}\{W(t)-tW(1)\}dt + o_p(k^{-1/2})\big) \\
&+ (1-\alpha T_n)\frac{1}{\alpha^2(\beta^*-\alpha)^2}\big(\frac{\alpha(\beta^*)^2(\beta^*-\alpha)^2}{2\beta^*-\alpha} + O_p(\epsilon_n)\big)\Big\} \\
=&-\frac{(\beta^*)^2}{\alpha^2}\int_0^1 t^{\beta^*/\alpha-2}\{W(t)-tW(1)\}dt + o_p(1).
\end{aligned}$$

Proof of Theorem 2.1. Define $l_1(\gamma) = k^{-1}\log L_1(\gamma)$,

$$\Delta = (\gamma-\gamma_0)\otimes(a^{-1}(n/k),1,1,1,1)^T,$$

$$M_1(\gamma) = (a(n/k),1,1,1,1)^T \otimes \frac{\partial}{\partial\gamma} l_1(\gamma),$$

$$M_2(\gamma) = \begin{pmatrix} a^2(n/k) & a(n/k) & a(n/k) & a(n/k) & a(n/k) \\ a(n/k) & 1 & 1 & 1 & 1 \\ a(n/k) & 1 & 1 & 1 & 1 \\ a(n/k) & 1 & 1 & 1 & 1 \\ a(n/k) & 1 & 1 & 1 & 1 \end{pmatrix} \otimes \frac{\partial^2}{\partial\gamma^T\partial\gamma} l_1(\gamma),$$

where $\otimes$ is the Kronecker product. It follows from Taylor expansion that

$$l_1(\gamma) - l_1(\gamma_0) = \Delta^T M_1(\gamma_0) + \frac{1}{2}\Delta^T M_2(\bar\gamma)\Delta,$$

where $\bar\gamma = (\bar\gamma_1,\cdots,\bar\gamma_5)^T$ and $\bar\gamma_i$ lies between γ_{i0} and γ_i for $i=1,\cdots,5$. Like the proofs in Lemmas 4.2–4.4, Lemma 4.1 can be used to show that

$$M_1(\gamma_0) \xrightarrow{p} 0 \quad \text{and} \quad M_2(\gamma_0) \xrightarrow{p} -M,$$

where M is a positive definite 5×5 matrix. So, the smallest eigenvalue of M, say λ_1, is positive. Using Lemma 4.1 again, we can further show that there exists $\delta_0 > 0$ such that $M_2(\gamma) \geq M - \lambda_1/2$ with probability tending to one when γ satisfies $||\gamma-\gamma_0|| \leq \delta_0$. Therefore

$$\Delta^T M_2(\bar\gamma)\Delta \geq \Delta^T M\Delta - \frac{\lambda_1}{2}\Delta^T\Delta \geq \frac{\lambda_1}{2}\delta_0^2$$

with probability tending to one when $||\gamma - \gamma_0|| = \delta_0$. Thus, with probability tending to one, $l_1(\gamma) < l_1(\gamma_0)$ for all λ in a small ϵ-sphere about γ_0. To complete the proof, just proceed as in the proof of Theorem 6.4.1 in Lehmann (1983).

Proof of Theorem 2.2. The theorem follows from (4.12), Lemmas 4.4 and 4.5.

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