Empirical Likelihood Method For Intermediate Quantiles

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Abstract

Intermediate quantiles play an important role in the statistics of extremes with particular applications in risk management. For interval estimation of quantiles, Chen and Hall (1993) proposed the so-called smoothed empirical likelihood method. In this paper, we apply the method in Chen and Hall (1993) to construct confidence intervals for an intermediate quantile and show that the choice of the bandwidth is dramatically different from that in Chen and Hall (1993).

Keywords: Confidence interval; Empirical likelihood; Intermediate quantile

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1 Introduction

Suppose $X_1, \cdots, X_n$ are independent and identically distributed random variables with distribution function F. The q-th quantile of F is defined as $F^-(q)$, where F^- denotes the inverse function of F. Quantiles are of importance in statistical inference, and both empirical quantile estimation and kernel smooth quantile estimation have been studied in the literature for a long history. Interval estimation for a quantile includes the normal approximation method, the bootstrap method, the Jackknife method and the empirical likelihood method.

The empirical likelihood method introduced by Owen (1988, 1990) is a powerful non-parametric method for constructing confidence regions, which has been extended and applied in many different settings. We refer to Owen (2001) for a comprehensive overview. Chen and Hall (1993) proposed the so-called smoothed empirical likelihood method to construct confidence intervals for the q-th quantile and showed that smoothing is necessary to achieve Bartlett correction. Smoothing also makes the optimization in the empirical likelihood method easy in genenal.

In this paper, we investigate the feasibility of applying the method in Chen and Hall (1993) to an intermediate quantile. When $q = q_n \to 1$ and $n(1 - q_n) \to \infty$ as $n \to \infty$, we call $F^{-}(q_n)$ an intermediate quantile. Intermediate quantiles play an important role in the statistics of extremes with particular applications to risk management. For example, Pickands (1975) used three intermediate order statistics to estimate the extreme value index; Viharos (1997) employed a linear combination of intermediate order statistics to estimate the tail index of a heavy tailed distribution; Csörgő and Steinebach (1991) applied intermediate order statistics to estimate the adjustment coefficient in risk theory. More references on the study of intermediate quantiles can be found in Peng and Yang (2009).

Because the intermediate quantile $F^{-}(q_n)$ tends to the right endpoint of the underlying distribution function F, some conditions on the tail behavior of F are needed in order to derive the asymptotic limit of the empirical intermediate quantile. Extreme value conditions are employed for such a study; see Dekkers and de Haan (1989), Dekkers, Einmahl and de Haan (1989) and Csörgő and Horváth (1993). In this paper, using extreme value conditions, we show that the method in Chen and Hall (1993) is applicable to an intermediate quantile, but the choice of bandwidth depends on the tail behavior, which is dramatically different from the bandwidth conditions in Chen and Hall (1993).

We organize this paper as follows. In Section 2, the method and main results are presented. All proofs are put in Section 3.

2 Method and main results

Throughout we assume that $K(x)$ is a symmetric density with support in $[-1, 1]$. Put $G(x) = \int_{-\infty}^{x} K(y)dy$. As in Chen and Hall (1993), we define the smoothed empirical likelihood for the q_n-th intermediate quantile $\theta_n = F^{-}(q_n)$ as

$$L_n(q_n, \theta_n) = \sup\{\prod_{i=1}^{n} p_i : \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i G(\frac{\theta_n - X_i}{h}) = q_n, p_i > 0, i = 1, \dots, n\},$$

where $h = h(n) > 0$ is a bandwidth.

Define $w_i = w_i(\theta_n) = G((\theta_n - X_i)/h) - q_n$. Since $\prod_{i=1}^{n} p_i$ attains its maximum at $p_i = 1/n$, the empirical likelihood ratio at θ_n is defined as

$$R_n(q_n, \theta_n) = \sup\{\prod_{i=1}^{n}(np_i) : \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i w_i = 0, p_i > 0, i = 1, \dots, n\}.$$

By applying the standard method of Lagrange multiplier, we know that $R_n(q_n, \theta_n)$ attains its maximum at

$$p_i = \frac{1}{n}\frac{1}{1 + \lambda w_i},$$

where $\lambda = \lambda(\theta_n)$ is determined by

$$\frac{1}{n}\sum_{i=1}^{n} \frac{w_i}{1 + \lambda w_i} = 0. \tag{1}$$

Thus, the empirical log likelihood ratio is given by

$$l_n(q_n, \theta_n) = 2\sum_{i=1}^{n} \log(1 + \lambda w_i). \tag{2}$$

In order to derive the asymptotic properties of $l_n(q_n, \theta_n)$, we employ the following von Mises' condition:

$$\lim_{t \to \theta^*} \frac{\{1 - F(t)\}F''(t)}{\{F'(t)\}^2} = -\gamma - 1 \tag{3}$$

for some $\gamma \in R$, where $\theta^* = \sup\{x : F(x) < 1\}$. Note that the above condition implies that F lies in the domain of attraction of an extreme value distribution with index γ; see Theorem 1.1.8 of de Haan and Ferreira (2006).

Our main result is as follows.

Theorem 1. Assume that $q_n \to 1$ and $n(1-q_n) \to \infty$ as $n \to \infty$. Suppose $F''(x)$ exists and $F'(x)$ is positive for all x in some left neighborhood of θ^*. If (3) holds for some $\gamma \neq 0$ and

$$\begin{cases} n(1-q_n)\{\frac{h}{F^-(q_n)}\}^4 \to 0, \quad \text{when} \quad \gamma > 0, \\ n(1-q_n)\{\frac{h}{\theta^* - F^-(q_n)}\}^4 \to 0, \quad \text{when} \quad \gamma < 0, \end{cases} \tag{4}$$

as $n \to \infty$, then $l_n(q_n, \theta_n) \xrightarrow{\text{d}} \chi_1^2$ as $n \to \infty$.

Based on the above theorem, an empirical likelihood based confidence interval for θ_n with level α can be obtained as

$$I_\alpha(h, n) = \{\beta_n : l_n(q_n, \beta_n) \leq z_\alpha\},$$

where z_α is chosen to satisfy $P(\chi_1^2 \leq z_\alpha) = \alpha$.

Remark 1. When $\gamma > 0$, condition (4) does not imply that the bandwidth h has to tend to zero. This is dramatically different from the method for a fixed quantile. However, when $\gamma < 0$, condition (4) does imply that $h \to 0$ since $n(1-q_n) \to \infty$ implies that $h/\{\theta^* - F^-(q_n)\} \to 0$ as $n \to \infty$.

Remark 2. Note that (3) implies that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma \quad \text{when} \quad \gamma > 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{U(\infty) - U(tx)}{U(\infty) - U(t)} = x^\gamma \quad \text{when} \quad \gamma < 0.$$

Hence, as n large enough, we have, for any $\epsilon > 0$,

$$F^-(q_n) = U(\frac{1}{1-q_n}) \geq (\frac{1}{1-q_n})^{\gamma-\epsilon} \quad \text{when} \quad \gamma > 0,$$

and

$$\theta^* - F^-(q_n) = U(\infty) - U(\frac{1}{1-q_n}) \geq (\frac{1}{1-q_n})^{\gamma-\epsilon} \quad \text{when} \quad \gamma < 0.$$

Therefore, a sufficient condition for (4) is

$$\lim_{n\to\infty} nh^4(1-q_n)^{4\gamma-4\epsilon+1} = 0 \tag{5}$$

for some $\epsilon > 0$. Condition (5) can be employed to choose the bandwidth via estimating the extreme value index γ.

3 Proofs

Throughout we define $\bar{w}_j = \frac{1}{n}\sum_{i=1}^n w_i^j$ and $\mu_j = E(\bar{w}_j)$ for $j = 1, 2$. First we show the following lemma.

Lemma 1. Under conditions of Theorem 1, we have, as $n \to \infty$,

$$\begin{cases} \mu_1 = \frac{h^2}{2}F''(\theta_n)(1+o(1))\int_{-1}^1 z^2K(z)dz, \quad \text{when} \quad \gamma \neq -1, \\ \mu_1 = o(\frac{h^2(1-q_n)}{\{\theta^*-F^-(q_n)\}^2}), \quad \text{when} \quad \gamma = -1, \\ \mu_2 = (1-q_n)(1+o(1)), \\ \frac{n\mu_1^2}{\mu_2} \to 0. \end{cases}$$

Proof. Let $U(t)$ denote the inverse of $1/\{1-F(x)\}$. Then Corollary 1.1.10 of de Haan and Ferreira (2006) says that (3) implies that

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{tU'(t)} = \frac{x^\gamma - 1}{\gamma} \quad \text{for all} \quad x > 0. \tag{6}$$

Applying Theorem B.2.2 of de Haan and Ferreira (2006) to (6), we have

$$\begin{cases} \lim_{t\to\infty} \frac{tU'(t)}{U(t)} = \gamma, \quad \text{when} \quad \gamma > 0, \\ \lim_{t\to\infty} \frac{tU'(t)}{U(\infty)-U(t)} = -\gamma, \quad \text{when} \quad \gamma < 0. \end{cases} \tag{7}$$

By (7), the proofs of Theorems 1.1.11 and 1.1.13 of de Haan and Ferreira (2006), we have

$$\begin{cases} \lim_{t\to\infty} \frac{tF'(t)}{1-F(t)} = \gamma^{-1}, \quad \text{when} \quad \gamma > 0, \\ \lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma} \quad \text{for all} \quad x > 0, \quad \text{when} \quad \gamma > 0, \\ \lim_{t\to\theta^*} \frac{(\theta^*-t)F'(t)}{1-F(t)} = -\gamma^{-1}, \quad \text{when} \quad \gamma < 0, \\ \lim_{t\to 0} \frac{1-F(\theta^*-tx)}{1-F(\theta^*-t)} = x^{-1/\gamma}, \quad \text{for all} \quad x > 0, \quad \text{when} \quad \gamma < 0. \end{cases} \tag{8}$$

It follows from Taylor's expansion that

$$\begin{aligned}
\mu_1 = E\bar{w}_1 &= E\{G(\frac{\theta_n - X_i}{h}) - q_n\} \\
&= \int_{-\infty}^{\infty} G(\frac{\theta_n - x}{h}) dF(x) - q_n \\
&= \int_{-1}^{1} F(\theta_n - hz)K(z)dz - q_n \\
&= \int_{-1}^{1} \{F(\theta_n) - hzF'(\theta_n) + \frac{h^2 z^2}{2} F''(\theta_n^*(z))\} K(z)dz - q_n \\
&= \frac{h^2}{2} \int_{-1}^{1} z^2 K(z) F''(\theta_n^*(z)) dz, \qquad (9)
\end{aligned}$$

where $\theta_n^*(z) \in (\theta_n - h, \theta_n + h)$ for all $z \in [-1, 1]$. Note that conditions in Theorem 1 imply that

$$\begin{cases} \lim_{n\to\infty} \frac{h}{F^-(q_n)} = 0 \quad \text{when} \quad \gamma > 0, \\ \lim_{n\to\infty} \frac{h}{\theta^* - F^-(q_n)} = 0 \quad \text{when} \quad \gamma < 0, \end{cases} \qquad (10)$$

which implies that

$$\begin{cases} \lim_{n\to\infty} \frac{\theta_n^*(z)}{\theta_n} = 1 \quad \text{uniformly in} \quad z \in [-1, 1] \quad \text{when} \quad \gamma > 0, \\ \lim_{n\to\infty} \frac{\theta^* - \theta_n^*(z)}{\theta^* - \theta_n} = 1 \quad \text{uniformly in} \quad z \in [-1, 1] \quad \text{when} \quad \gamma < 0. \end{cases} \qquad (11)$$

By (3), (8) and (11), we have

$$\begin{cases} \lim_{n\to\infty} F''(\theta_n^*(z))/F''(\theta_n) = 1 \quad \text{uniformly in} \quad z \in [-1, 1] \quad \text{when} \quad \gamma \neq -1, \\ F''(\theta_n^*(z)) = o(\frac{1-q_n}{\{\theta^* - F^-(q_n)\}^2}) \quad \text{uniformly in} \quad z \in [-1, 1] \quad \text{when} \quad \gamma = -1, \\ F''(\theta_n) \sim \frac{-\gamma-1}{\gamma^2} \frac{1-q_n}{\{F^-(q_n)\}^2} \quad \text{when} \quad \gamma > 0, \\ F''(\theta_n) \sim \frac{-\gamma-1}{\gamma^2} \frac{1-q_n}{\{\theta^* - F^-(q_n)\}^2} \quad \text{when} \quad \gamma < 0 \quad \text{and} \quad \gamma \neq -1, \\ F''(\theta_n) = o(\frac{\{F'(\theta_n)\}^2}{1-F(\theta_n)}) \quad \text{when} \quad \gamma = -1. \end{cases} \qquad (12)$$

It follows from (9) and (12) that

$$\begin{cases} \mu_1 = \frac{h^2}{2} F''(\theta_n)(1 + o(1)) \int_{-1}^{1} z^2 K(z) dz \quad \text{when} \quad \gamma \neq -1, \\ \mu_1 = o(\frac{h^2(1-q_n)}{\{\theta^* - F^-(q_n)\}^2}) \quad \text{when} \quad \gamma = -1. \end{cases} \qquad (13)$$

Write

$$\begin{aligned}
\mu_2 = E\bar{w}_2 &= E\left\{G\left(\frac{\theta_n - X_1}{h}\right) - q_n\right\}^2 \\
&= E\left\{G^2\left(\frac{\theta_n - X_1}{h}\right)\right\} - 2q_n E\left\{G\left(\frac{\theta_n - X_1}{h}\right)\right\} + q_n^2. \qquad (14)
\end{aligned}$$

By Taylor's expansion, we have

$$\begin{aligned}
& E\left\{G^2\left(\tfrac{\theta_n - X_1}{h}\right)\right\} \\
= & \int_{-\infty}^{\infty} G^2\left(\tfrac{\theta_n - x}{h}\right) dF(x) \\
= & \tfrac{2}{h}\int_{-\infty}^{\infty} F(x) G\left(\tfrac{\theta_n - x}{h}\right) K\left(\tfrac{\theta_n - x}{h}\right) dx \\
= & 2\int_{-1}^{1} F(\theta_n - hz) G(z) K(z) dz \\
= & 2\int_{-1}^{1}\{F(\theta_n) - hzF'(\theta_n) + \tfrac{h^2z^2}{2}F''(\theta_n^*(z))\}K(z)G(z)dz \\
= & F(\theta_n) - 2hF'(\theta_n)\int_{-1}^{1} zK(z)G(z)dz + h^2\int_{-1}^{1} z^2F''(\theta_n^*(z))K(z)G(z)dz \\
= & q_n - 2hF'(\theta_n)\int_{-1}^{1} zK(z)G(z)dz + h^2\int_{-1}^{1} z^2F''(\theta_n^*(z))K(z)G(z)dz.
\end{aligned} \tag{15}$$

Using (12), (13), (14) and (15), we have

$$\begin{aligned}
\mu_2 \;=\; & q_n - q_n^2 - 2hF'(\theta_n)\int_{-1}^{1} zK(z)G(z)dz \\
& + h^2F''(\theta_n)(1+o(1))\{\int_{-1}^{1} z^2K(z)G(z)dz - q_n\int_{-1}^{1} z^2K(z)\,dz\}
\end{aligned} \tag{16}$$

when $\gamma \neq -1$, and

$$\mu_2 = q_n - q_n^2 - 2hF'(\theta_n)\int_{-1}^{1} zK(z)G(z)dz + o\left(\frac{h^2(1-q_n)}{\{\theta^* - F^-(q_n)\}^2}\right) \tag{17}$$

when $\gamma = -1$. By (8), (10) and (12), we have

$$\left\{\begin{array}{l}
\frac{hF'(\theta_n)}{1-q_n} \sim \gamma^{-1}\frac{h}{F^-(q_n)} = o(1) \quad \text{when} \quad \gamma > 0, \\
\frac{hF'(\theta_n)}{1-q_n} \sim -\gamma^{-1}\frac{h}{\theta^* - F^-(q_n)} = o(1) \quad \text{when} \quad \gamma < 0, \\
\frac{h^2F''(\theta_n)}{1-q_n} \sim \frac{-\gamma-1}{\gamma^2}\{\frac{h}{F^-(q_n)}\}^2 = o(1) \quad \text{when} \quad \gamma > 0, \\
\frac{h^2F''(\theta_n)}{1-q_n} \sim \frac{-\gamma-1}{\gamma^2}\{\frac{h}{\theta^* - F^-(q_n)}\}^2 = o(1) \quad \text{when} \quad \gamma < 0 \quad \text{and} \quad \gamma \neq -1, \\
\frac{h^2F''(\theta_n)}{1-q_n} = o(\{\frac{h}{\theta^* - F^-(q_n)}\}^2) = o(1) \quad \text{when} \quad \gamma = -1.
\end{array}\right. \tag{18}$$

It follows from (12), (13), (16), (17), (18) and conditions in Theorem 1 that

$$\left\{\begin{array}{l}
\mu_2 = (1-q_n)(1+o(1)), \\
\frac{n\mu_1^2}{\mu_2} = o(1).
\end{array}\right. \tag{19}$$

Hence, the lemma follows from (13) and (19). ■

Proof of Theorem 1. Observe that

$$\begin{aligned} 0 &= n^{-1}|\sum_{i=1}^{n} w_i(1+\lambda w_i)^{-1}| \\ &= n^{-1}|\sum_{i=1}^{n}\{\lambda w_i^2(1+\lambda w_i)^{-1} - w_i\}| \\ &\geq |\lambda|(1+|\lambda|)^{-1}\bar{w}_2 - |\bar{w}_1|, \end{aligned}$$

since $\max_{1\leq i\leq n}|w_i| \leq 1$. Thus,

$$|\lambda|(1-\frac{|\bar{w}_1|}{\bar{w}_2}) \leq \frac{|\bar{w}_1|}{\bar{w}_2}. \tag{20}$$

By Lemma 1, the law of large numbers and the central limit theorem, we have

$$\begin{aligned} \frac{\bar{w}_1}{\bar{w}_2} &= \frac{\frac{\sqrt{n}(\bar{w}_1-\mu_1)}{\sqrt{\mu_2-\mu_1^2}} + \frac{\sqrt{n}\mu_1}{\sqrt{\mu_2-\mu_1^2}}}{\frac{\sqrt{n}\mu_2}{\sqrt{\mu_2-\mu_1^2}}\frac{\bar{w}_2}{\mu_2}} \\ &= O_p(\frac{\sqrt{\mu_2-\mu_1^2}}{\sqrt{n}\mu_2}) \\ &= O_p(\frac{1}{\sqrt{n(1-q_n)}}). \end{aligned}$$

Hence, it follows from (20) that

$$\lambda = O_p(\frac{1}{\sqrt{n(1-q_n)}}). \tag{21}$$

Now

$$\begin{aligned} 0 &= \frac{1}{n}\sum_{i=1}^{n}\frac{w_i}{1+\lambda w_i} \\ &= \frac{1}{n}\sum_{i=1}^{n} w_i\{1-\lambda w_i + \frac{(\lambda w_i)^2}{1+\lambda w_i}\} \\ &= \bar{w}_1 - \lambda\bar{w}_2 + n^{-1}\sum_{i=1}^{n}\frac{\lambda^2 w_i^3}{1+\lambda w_i} \\ &= \bar{w}_1 - \lambda\bar{w}_2 + O_p(\lambda^2\bar{w}_2). \end{aligned}$$

Thus,

$$\lambda = \bar{w}_1\bar{w}_2^{-1} + O_p(\frac{1}{n(1-q_n)}). \tag{22}$$

Applying Taylor's expansion to (2) and using (22), we can show that

$$\begin{aligned} l_n(q_n, \theta_n) &= 2\sum_{i=1}^n \log(1+\lambda w_i) \\ &= 2n\lambda\bar{w}_1 - n\lambda^2\bar{w}_2 + 2\sum_{i=1}^n \eta_i \\ &= n\bar{w}_1^2\bar{w}_2^{-1} - n\bar{w}_2 O_p(\frac{1}{n^2(1-q_n)^2}) + 2\sum_{i=1}^n \eta_i \\ &= \frac{\{\frac{\sqrt{n}(\bar{w}_1-\mu_1)}{\sqrt{\mu_2-\mu_1^2}} + \frac{\sqrt{n}\mu_1}{\sqrt{\mu_2-\mu_1^2}}\}^2}{\frac{\mu_2}{\mu_2-\mu_1^2}\frac{\bar{w}_2}{\mu_2}} - n\bar{w}_2 O_p(\frac{1}{n^2(1-q_n)^2}) + 2\sum_{i=1}^n \eta_i, \end{aligned} \tag{23}$$

where

$$P(|\eta_i| \le C|\lambda w_i|^3, 1 \le i \le n) \to 1 \quad \text{as} \quad n \to \infty$$

for some constant $C > 0$. By (21) and Lemma 1, we have

$$2\sum_{i=1}^n \eta_i = O_p(|\lambda|^3 \sum_{i=1}^n w_i^2) = O_p(\frac{1}{\sqrt{n(1-q_n)}}). \tag{24}$$

Hence, Theorem 1 follows from Lemma 1, (21), (23), (24) and the central limit theorem.■

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