Goodness-of-fit test for two-parameter Weibull distributions

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Summary

In this paper, new goodness-of-fit tests are proposed to testing the two-parameter Weibull distribution. Unlike existing methods, the asymptotic limits of the new tests are both independent of the underlying Weibull distribution and invariant to the plugged parameter estimator as long as the estimator has the standard rate of convergence. Hence, critical points can be simulated from either the limits or the test statistics by sampling from any one of Weibull distributions.

Key words: Asymptotic limit, Gaussian process, goodness-of-fit test, Gumbel distribution, two-parameter Weibull distribution

1 Introduction

The two-parameter Weibull distribution is defined as

$$F(x;\gamma,\theta)=1-\exp\{-(x/\theta)^{\gamma}\} \tag{1.1}$$

for $x>0$, where $\theta>0$ and $\gamma>0$ are called scale parameter and shape parameter, respectively. This class of distributions includes the exponential distribution as a special case, and plays an important role in modeling lifetimes of devices. For estimating θ and γ, various different estimators have been proposed in the literature such as ordinary least squares estimation, generalized least squares estimation, maximum likelihood estimation, moment estimator, etc.; see Bain and Antle (1967), Cohen (1965), Engelhardt and Bain (1974), Menon (1963), White (1965). A comparison on different estimators is given by Al-Baidhani and Sinclair (1987). Interval estimation for the parameters and other quantities related with the two-parameter Weibull distribution can be found in Chen (1998) and Yang,

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Xie and Wong (2007). Inference for two independent two-parameter Weibull distributions is given in Wu, Wong and Ng (2005).

Some goodness-of-fit tests for the two-parameter Weibull distribution have been proposed in the literature too. Shapiro and Brain (1987) proposed the so called W-test. Although this W-test avoids estimating the parameters θ and γ, the null distribution of this test is not analytically tractable and it remains unknown whether the limit distribution exists. Coles (1989) proposed a test via the stabilized probability plot, which involves estimating θ and γ. Khamis (1997) proposed the δ-corrected Kolmogorov-Smirnov test, where the MLE for θ and γ was employed. All these three papers proposed to obtain critical points via simulating the test statistic for each sample size. Recently, Cabana and Quiroz (2005) proposed to employ the empirical moment generating function and affine invariant estimators for θ and γ such as moment estimators. Since the asymptotic limits of the above mentioned tests depend on the employed estimators for θ and γ, one has to recalculate the critical points for different estimators. In this paper, we propose some new tests whose limits are independent of parameters and remain the same for any plugged parameter estimator as long as it has $n^{-1/2}$ rate of convergence. Hence, for obtaining critical points, one can either simulate the limiting distribution or simulate random samples from a particular Weibull distribution, say $(\theta, \gamma) = (1, 1)$, and then simulate the test statistics with the same (θ, γ) instead of estimators from the simulated samples. Indeed, one can simply simulate critical points via uniform random variables; see Section 3 for details.

We organize this paper as follows. Section 2 presents the new method and the asymptotic limits of the proposed test statistics. A simulation study is given in Section 3. Proofs are put in an appendix.

2 Methodology

Let $X, X_1, \cdots, X_n$ be independent and identically distributed positive random variables with continuous distribution function F. Define

$$G(x) = P(\log X \leq x), \quad \bar{G}(x) = 1 - G(x), \quad \bar{G}^*(x) = -\log \bar{G}(x).$$

First we show the following proposition.

Proposition 1. The following two statements are equivalent:

i) F satisfies (1.1);

ii)

$$\{\bar{G}(y)\}^{\bar{G}^*(z)} = \{\bar{G}(y+z-x)\}^{\bar{G}^*(x)} \quad \text{for all} \quad x,y,z \in R. \tag{2.1}$$

The above equation (2.1) motivates us to consider the following tests. Define

$$G_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(\log X_i \le x), \quad \bar{G}_n(x) = 1 - G_n(x), \quad \bar{G}_n^*(x) = -\log \bar{G}_n(x),$$

$$\Delta_n(x,y,z) = \left(\{\bar{G}_n(y+z-x)\}^{\bar{G}_n^*(x)} - \{\bar{G}_n(y)\}^{\bar{G}_n^*(z)}\right) \min\{\bar{G}(y+z-x), \bar{G}(y), \bar{G}(z), \bar{G}(x)\},$$

$$\begin{aligned} \hat{\Delta}_n(x,y,z) &= \left(\{\bar{G}_n(y+z-x)\}^{\bar{G}_n^*(x)} - \{\bar{G}_n(y)\}^{\bar{G}_n^*(z)}\right) \\ &\quad \times \exp\{-\hat{\theta}^{-\hat{\gamma}} \max(e^{\hat{\gamma}x}, e^{\hat{\gamma}(y+z-x)}, e^{\hat{\gamma}y}, e^{\hat{\gamma}z})\}, \end{aligned}$$

where $(\hat{\theta}, \hat{\gamma})$ is an estimator of (θ, γ) with the $n^{-1/2}$ rate of convergence. Then our proposed test statistics are defined as

$$T_1 = \sqrt{n} \sup_{x,y,z} |\hat{\Delta}_n(x,y,z)|$$

and

$$\begin{aligned} T_2 &= -n \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\Delta}_n^2(x,y,z)\, d\exp\{-\hat{\theta}^{-\hat{\gamma}}(e^{\hat{\gamma}x} + e^{\hat{\gamma}y} + e^{\hat{\gamma}z})\} \\ &= n \textstyle\int_0^1\int_0^1\int_0^1 \hat{\Delta}_n^2\big(\frac{\log(-\hat{\theta}^{\hat{\gamma}}\log x)}{\hat{\gamma}}, \frac{\log(-\hat{\theta}^{\hat{\gamma}}\log y)}{\hat{\gamma}}, \frac{\log(-\hat{\theta}^{\hat{\gamma}}\log z)}{\hat{\gamma}}\big)\, dxdydz. \end{aligned}$$

The asymptotic limits of these tests are given as follows.

Theorem 1. Suppose F satisfies (1.1). Then, as $n \to \infty$,

$$\begin{aligned} T_1 \;\; &\xrightarrow{d} \sup\nolimits_{0\le x,y,z\le 1} \big| W(1-y^{\log z/\log x}) y^{-\log z} \log(x) y^{-\log z/\log x} \min(x,y,z,y^{\log z/\log x}) \\ &+W(1-x) y^{-\log z} \tfrac{\log(y)\log(z)}{\log(x)} x^{-1} \min(x,y,z,y^{\log z/\log x}) \\ &-W(1-y) y^{-\log z} \log(z) y^{-1} \min(x,y,z,y^{\log z/\log x}) \\ &-W(1-z) y^{-\log z} \log(y) z^{-1} \min(x,y,z,y^{\log z/\log x}) \big| \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} T_2 \;\; &\xrightarrow{d} \textstyle\int_0^1\int_0^1\int_0^1 \{W(1-y^{\log z/\log x}) y^{-\log z} \log(x) y^{-\log z/\log x} \min(x,y,z,y^{\log z/\log x}) \\ &+W(1-x) y^{-\log z} \tfrac{\log(y)\log(z)}{\log(x)} x^{-1} \min(x,y,z,y^{\log z/\log x}) \\ &-W(1-y) y^{-\log z} \log(z) y^{-1} \min(x,y,z,y^{\log z/\log x}) \\ &-W(1-z) y^{-\log z} \log(y) z^{-1} \min(x,y,z,y^{\log z/\log x})\}^2\, dxdydz. \end{aligned} \tag{2.3}$$

where $W(x)$ is a Brownian bridge.

Remark 1. Since the logarithm of a gamma random variable follows from a Gumbel distribution, the proposed method can easily be applied to testing whether a distribution belongs to the class of Gumbel distributions.

3 Simulation study

In general, a test based on an integrated distance is more powerful than that based on the supremum distance. Hence, we focus on the study of T_2. It follows from the proof of Theorem 1 that the test statistic T_2 has the same asymptotic limit as

$$\begin{aligned}\tilde{T}_2 \quad &= n\int_0^1\int_0^1\int_0^1 \big(\exp\{-\log\bar{H}_n(1-x)\log\bar{H}_n(1-y^{\log z/\log x})\} \\ &\quad -\exp\{-\log\bar{H}_n(1-z)\log\bar{H}_n(1-y)\}\big)^2\min\{x^2,y^2,z^2,\exp(2\log z\log y/\log x)\}\,dxdydz,\end{aligned}$$

where $1-\bar{H}_n(x)$ is the empirical distribution of a random sample of size n from the uniform distribution on $[0,1]$. Hence, for obtaining critical points of T_2, we compute $\tilde{T}_2$ by drawing $100,000$ random samples from the uniform distribution on $[0,1]$ and denote by $\tilde{T}_{cri}$. We also draw $100,000$ random samples from $F(x;1,1)$ in (1.1) and calculate Cramer-von Mises test statistic (see D'Agostino and Stephens (1996)) and T_2 with either the maximum likelihood estimator or moment estimator for (θ,γ) to obtain critical points. Let's denote these critical points by $CM_{cri,MLE}$, $CM_{cri,MM}$, $T_{cri,MLE}$ and $T_{cri,MM}$. For computing T_2 and $\tilde{T}_2$, we use Monte Carlo integration method by drawing $20,000$ random samples from the uniform distribution on $[0,1]^3$. In Tables 1 and 2, we report these critical points with levels 0.10 and 0.05 for sample size $n=20,50,100,200,500,1000$. From these two tables, we observe that critical points for the new test T_2 are not dependent on the employed estimators.

In order to study the power of the proposed test T_2 for a small sample size, we draw 20,000 random samples with sample size $n=20,50,100$ from $Beta(0.25,0.25), Beta(2,2), N(0,1), TruncN(-1,1)$, where $TruncN(a,b)$ denotes the standard normal truncated to the left at a and to the right at b. Using Table 2, we calculate the empirical powers for Cramer-von Mises test and T_2 with maximum likelihood estimators for the parameters at level 0.05 in Table 3.

Next we draw $20,000$ random samples of size $n=1000$ from the mixture distributions $(1-\rho)F(x;\theta,\gamma)+\rho TruncN(a,b)$, where $\rho\in[0,1]$ and F is given in (1.1). Particularly we consider $\rho=0.1,0.3,0.5,0.7$, $\theta=\gamma=2$, $(a,b)=(-2,1),(-1,1),(-1,2)$ and compute the powers of Cramer-von Mises test and T_2 with either maximum likelihood estimators or moment estimators for the parameters by using the critical points $CM_{cri,MLE}, CM_{cri,MM}, \tilde{T}_{cri}$ for $n = 1000$ given in Table 2. These powers are reported in Table 4, which shows that the limit of the new test statistic is independent of employed estimators for the parameters.

In summary, the proposed test statistic is comparable to Cramer-von Mises test statistic in term of powers, but unlike other existing methods, the proposed test has a limiting distribution independent of external parameter estimation.

Appendix: Proofs

Proof of Proposition 1. If (1.1) holds, then

$$\bar{G}(x) = \exp\{-\theta^{-\gamma}e^{\gamma x}\} \quad \text{and} \quad \bar{G}^*(x) = \theta^{-\gamma}e^{\gamma x},$$

which imply that

$$\{\bar{G}(y)\}^{\bar{G}^*(z)} = \{\bar{G}(y+z-x)\}^{\bar{G}^*(x)} = \exp\{-\theta^{-2\gamma}e^{\gamma(y+z)}\},$$

i.e., (2.1) holds.

If (2.1) holds, then taking $x = 0$ gives

$$\bar{G}^*(z)\bar{G}^*(y) = \bar{G}^*(0)\bar{G}^*(y+z)$$

for all $y, z \in R$. Hence $\bar{G}^*(x) = a_1 e^{a_2 x}$ for some $a_1, a_2 \in R$. Since $\bar{G}^*(x) > 0$ and it is an increasing function, we have $a_1 > 0$ and $a_2 > 0$. Thus, (1.1) holds for some positive constants θ and γ.

Proof of Theorem 1. Define $\bar{H}_n(1-x) = \bar{G}_n(\bar{G}^-(x))$, where G^- denotes the inverse function of G. It is easy to check that $H_n(x) = 1 - \bar{H}_n(x)$ has the same distribution as the empirical distribution function based on n independent random variables with uniform distribution function on $[0,1]$. Without loss of generality, we can assume that $H_n(x) = \frac{1}{n}\sum_{i=1}^n I(U_i \le x)$, where $U_1, \cdots, U_n$ are independent random variables with uniform distribution on $[0,1]$. Let $U_{n,1} \le \cdots \le U_{n,n}$ denote the order statistics of $U_1, \cdots, U_n$. Then

$$\sqrt{n}\{H_n(x) - x\} \xrightarrow{D} W(x) \tag{3.1}$$

in $D([0,1])$, where $W(x)$ is a Brownian bridge. Since $\bar{G}^-(x) = \log\theta + \gamma^{-1}\log(-\log x)$, we have

$$\bar{G}^-(y) + \bar{G}^-(z) - \bar{G}^-(x) = \bar{G}^-(y^{\log z/\log x}).$$

Hence

$$
\begin{aligned}
& \Delta_n(\bar{G}^-(x), \bar{G}^-(y), \bar{G}^-(z)) \\
= \; & \Big(\{\bar{H}_n(1-y^{\log z/\log x})\}^{-\log \bar{H}_n(1-x)} - \{\bar{H}_n(1-y)\}^{-\log \bar{H}_n(1-z)}\Big) \min(x,y,z,y^{\log z/\log x}) \\
= \; & \Big(\{\bar{H}_n(1-y^{\log z/\log x})\}^{-\log \bar{H}_n(1-x)} - \{y^{\log z/\log x}\}^{-\log \bar{H}_n(1-x)}\Big) \min(x,y,z,y^{\log z/\log x}) \\
& + \Big(\{y^{\log z/\log x}\}^{-\log \bar{H}_n(1-x)} - \{y^{\log z/\log x}\}^{-\log x}\Big) \min(x,y,z,y^{\log z/\log x}) \\
& + \Big(y^{-\log \bar{H}_n(1-z)} - \{\bar{H}_n(1-y)\}^{-\log \bar{H}_n(1-z)}\Big) \min(x,y,z,y^{\log z/\log x}) \\
& + \Big(y^{-\log z} - y^{-\log \bar{H}_n(1-z)}\Big) \min(x,y,z,y^{\log z/\log x}) \\
= \; & I_1(x,y,z) + I_2(x,y,z) + I_3(x,y,z) + I_4(x,y,z).
\end{aligned}
$$

It is known that

$$
\sup_{0<x\le 1} \frac{|\sqrt{n}\{\bar{H}_n(1-x)-x\}|}{x^{1/4}} = O_p(1) \quad \text{and} \quad \sup_{U_{n,1}\le x\le 1} \frac{x}{\bar{H}_n(1-x)} = O_p(1) \tag{3.2}
$$

(see Shorack and Wellner (1986)). It follows from (3.2) that

$$
\sup_{n^{-5/9}\le x\le 1} \Big|\frac{\bar{H}_n(1-x)-x}{x}\Big| = \sup_{n^{-5/9}\le x\le 1} \Big|\frac{\sqrt{n}\{\bar{H}_n(1-x)-x\}}{x^{1/4}} n^{-1/2} x^{-3/4}\Big| = o_p(1). \tag{3.3}
$$

Using (3.3) and the fact that

$$
|\log(1+x)-x| \le 2x^2 \quad \text{for all} \quad x \ge -1/2, \tag{3.4}
$$

we have

$$
\begin{aligned}
& \sup_{n^{-5/9}\le x\le 1} \sqrt{n}|x\{\log \bar{H}_n(1-x) - \log x\} - \{\bar{H}_n(1-x)-x\}| \\
= \; & \sup_{n^{-5/9}\le x\le 1} \sqrt{n}|x \log\{1 + \tfrac{\bar{H}_n(1-x)-\log x}{x}\} - \{\bar{H}_n(1-x)-x\}| \\
\le \; & O_p\left(\sup_{n^{-5/9}\le x\le 1} 2\sqrt{n}\{\bar{H}_n(1-x)-x\}^2 x^{-1}\right) \\
\le \; & O_p\left(\{\sup_{n^{-5/9}\le x\le 1} 2(nx)^{-1/2}\} \sup_{n^{-5/9}\le x\le 1}\{\tfrac{\sqrt{n}(\bar{H}_n(1-x)-x)}{x^{1/4}}\}^2\right) \\
= \; & o_p(1).
\end{aligned} \tag{3.5}
$$

By (3.1), (3.5) and Taylor expansion, we have

$$
\begin{aligned}
& \sup_{0\le x,y\le 1, n^{-5/9}\le z\le 1} |\sqrt{n} I_4(x,y,z) + W(1-z) y^{-\log z}\log(y) z^{-1}\min(x,y,z,y^{\log z/\log x})| \\
= & \sup_{0\le x,y\le 1, n^{-5/9}\le z\le 1} |\sqrt{n}\{\log \bar{H}_n(1-z) - \log z\} y^{-\beta\log z - (1-\beta)\log \bar{H}_n(1-z)} \\
& \times \log(y)\min(x,y,z,y^{\log z/\log x}) + W(1-z) y^{-\log z}\log(y) z^{-1}\min(x,y,z,y^{\log z/\log x})| \\
= & \sup_{0\le x,y\le 1, n^{-5/9}\le z\le 1} |\sqrt{n}\{z(\log \bar{H}_n(1-z) - \log z) - (\bar{H}_n(1-z) - z)\} \\
& \times y^{-\beta\log z - (1-\beta)\log \bar{H}_n(1-z)}\log(y) z^{-1}\min(x,y,z,y^{\log z/\log x})| \\
& + \sup_{0\le x,y\le 1, n^{-5/9}\le z\le 1} |\{\sqrt{n}(\bar{H}_n(1-z) - z) + W(1-z)\} \\
& \times y^{-\beta\log z - (1-\beta)\log \bar{H}_n(1-z)}\log(y) z^{-1}\min(x,y,z,y^{\log z/\log x})| \qquad (3.6) \\
& + \sup_{0\le x,y\le 1, n^{-5/9}\le z\le 1} |W(1-z)\{y^{-\beta\log z - (1-\beta)\log \bar{H}_n(1-z)} - y^{-\log z}\} \\
& \times \log(y) z^{-1}\min(x,y,z,y^{\log z/\log x})| \\
= & O(\sup_{n^{-5/9}\le z\le 1} |\sqrt{n}\{z(\log \bar{H}_n(1-z) - \log z) - (\bar{H}_n(1-z) - z)\}|) \\
& + O_p(\sup_{n^{-5/9}\le z\le 1} |\sqrt{n}(\bar{H}_n(1-z) - z) + W(1-z)|) \\
& + O_p(\sup_{n^{-5/9}\le z\le 1} |W(z)|\{\beta_1(-\beta\log z + (1-\beta)\log \bar{H}_n(1-z)) + (1-\beta_1)\log z\}) \\
= & o_p(1),
\end{aligned}
$$

where $\beta = \beta(z) \in [0,1]$ and $\beta_1 = \beta_1(z) \in [0,1]$. It is easy to see that

$$
\sup_{0\le x,y,\le 1, 0\le z\le n^{-5/9}} |\sqrt{n} I_4(x,y,z)| \le \sup_{0\le z\le n^{-5/9}} \sqrt{n} z = o_p(1). \qquad (3.7)
$$

Hence, (3.6) and (3.7) imply that

$$
\sup_{0\le x,y,z\le 1} |\sqrt{n} I_4(x,y,z) + W(1-z) y^{-\log z}\log(y) z^{-1}\min(x,y,z,y^{\log z/\log x})| = o_p(1). \qquad (3.8)
$$

Using Taylor expansion, we have

$$
\begin{aligned}
& I_3(x,y,z) \\
= & \{\bar{H}_n(1-y) - y\} y^{-\log z}\log(z) y^{-1}\min(x,y,z,y^{\log z/\log x}) \\
& + \{\bar{H}_n(1-y) - y)\}\{y^{-\log \bar{H}_n(1-z)}\log(\bar{H}_n(1-z)) - y^{-\log z}\log z\} y^{-1}\min(x,y,z,y^{\log z/\log x}) \\
& - \tfrac{1}{2}\{\bar{H}_n(1-y) - y\}^2\{\beta y + (1-\beta)\bar{H}_n(1-y)\}^{-\log \bar{H}_n(1-z) - 2}\log(\bar{H}_n(1-z)) \times \\
& \quad \{\log(\bar{H}_n(1-z)) + 1\}\min(x,y,z,y^{\log z/\log x}) \\
= & II_1(x,y,z) + II_2(x,y,z) - II_3(x,y,z),
\end{aligned}
$$

where $\beta = \beta(y) \in [0,1]$. Further write

$$\begin{array}{rl} & II_3(x,y,z) \\ = & \frac{1}{2}\{\frac{\sqrt{n}(\bar{H}_n(1-y)-y)}{y^{1/4}}\}^2\{\beta y + (1-\beta)\bar{H}_n(1-y)\}^{-\log \bar{H}_n(1-y)} \\ & \times\{1+(1-\beta)\frac{\bar{H}_n(1-y)-y}{y}\}^{-2} n^{-1} y^{-1/2} y^{-1} \min(x,y,z,y^{\log z/\log x}). \end{array}$$

By (3.2), we have

$$\begin{array}{rl} & \sup_{n^{-5/9}\le y\le 1} |\frac{\bar{H}_n(1-y)-y}{y}| \\ = & \{\sup_{n^{-5/9}\le y\le 1} |\frac{\sqrt{n}(\bar{H}_n(1-y)-y)}{y^{1/4}}|\} \sup_{n^{-5/9}\le y\le 1}\{n^{-1/2}y^{-3/4}\} \\ = & o_p(1), \end{array}$$

and then

$$\sup_{0\le x,z\le 1, n^{-5/9}\le y\le 1} |\sqrt{n} II_3(x,y,z)| = O_p(\sup_{n^{-5/9}\le y\le 1} n^{-1/2}y^{-1/2}) = o_p(1). \tag{3.9}$$

Using Taylor expansion, we have

$$\begin{array}{rl} & II_2(x,y,z) \\ = & \{H_n(y)-y\}\{\log(\bar{H}_n(1-z)) - \log z\}\{y^{-\beta^*} - \beta^* y^{-\beta^*}\log y\} y^{-1}\min(x,y,z,y^{\log z/\log x}) \\ = & \{\bar{H}_n(1-y)-y\} z\{\log \bar{H}_n(1-z) - \log z\} y^{-\beta^*} y^{-1} z^{-1} \min(x,y,z,y^{\log z/\log x}) \\ & -\frac{\bar{H}_n(1-y)-y}{y^{1/4}} z\{\log \bar{H}_n(1-z) - \log z\}\beta^* y^{-\beta^*} y^{1/4}\log(y) y^{-1} z^{-1}\min(x,y,z,y^{\log z/\log x}), \end{array}$$

where $\beta^* = \beta^*(z)$ lies between $\log z$ and $\log \bar{H}_n(1-z)$. It follows from (3.3) that

$$\sup_{n^{-5/9}\le z\le 1} |\log \bar{H}_n(1-z) - \log z| = o_p(1),$$

i.e., $\sup_{n^{-5/9}\le z\le 1} |\beta^* - \log z| = o_p(1)$. Hence, (3.2) and (3.5) imply that

$$\sup_{0\le x,y\le 1, n^{-5/9}\le z\le 1} |\sqrt{n} II_2(x,y,z)| = o_p(1). \tag{3.10}$$

By (3.1), (3.9) and (3.10), we have

$$\sup_{0\le x\le 1, n^{-5/9}\le y,z\le 1} |\sqrt{n} I_3(x,y,z) + W(1-y) y^{-\log z}\log(z) y^{-1}\min(x,y,z,y^{\log z/\log x})| = o_p(1).$$

It is easy to see that

$$\sup_{0\le x,y,z\le 1,\min(y,z)\le n^{-5/9}} |\sqrt{n}I_3(x,y,z)| = \sup_{0\le \min(y,z)\le n^{-5/9}} \{\sqrt{n}\min(y,z)\} = o_p(1).$$

Therefore,

$$\sup_{0\le x,y,z\le 1} |\sqrt{n}I_3(x,y,z) + W(1-y)y^{-\log z}\log(z)y^{-1}\min(x,y,z,y^{\log z/\log x})| = o_p(1). \tag{3.11}$$

Similar to the proofs of (3.8) and (3.11), we can show that

$$\sup_{0\le x,y,z\le 1} |\sqrt{n}I_1(x,y,z) - W(1-y^{\log z/\log x})y^{-\log z}\log(x)y^{-\log z/\log x}\min(x,y,z,y^{\log z/\log x})| = o_p(1)$$

and

$$\sup_{0\le x,y,z\le 1} |\sqrt{n}I_2(x,y,z) - W(1-x)y^{-\log z}\frac{\log(y)\log(z)}{\log(x)}x^{-1}\min(x,y,z,y^{\log z/\log x})| = o_p(1),$$

i.e., $\sqrt{n}\sup_{x,y,z}|\Delta_n(x,y,z)|$ converges in distribution to the right hand side of (2.2). It is easy to verify that $\sqrt{n}\sup_{x,y,z}|\hat{\Delta}_n(x,y,z)|$ has the same limit as $\sqrt{n}\sup_{x,y,z}|\Delta_n(x,y,z)|$, i.e., (2.2) holds. Similarly, we can prove (2.3).

Acknowledgment. The author thanks an associate editor for helpful comments. Research was supported by NSF grant SES-0631608.

References

F.A. Al-Baidhani and C.D. Sinclair (1987). Comparison of methods of estimation of parameters of the Weibull distribution. *Commun. Statist. -Simula. 16, 373–384.*

L.J. Bain and C.E. Antle (1967). Estimation of parameters in the Weibull distribution. *Technometrics 9, 621–627.*

A. Cabana and A.J. Quiroz (2005). Using the empirical moment generating function in testing for the Weibull and the type I extreme value distributions. *Test 14, 417–431.*

Z. Chen (1998). Joint estimation for the parameters of Weibull distribution. *J. Statist. Plann.*

Inference 66, 113–120.

A.C. Cohen (1965). Maximum likelihood estimation in the Weibull distribution based on complete and censored samples. *Technometrics 7, 579–588.*

S.G. Coles (1989). On goodness-of-fit tests for the two-parameter Weibull distribution derived from the stabilized probability plot. *Biometrika 76, 593–598.*

R.B. D'Agostino and M.A. Stephens (1996). Goodness-of-Fit Techniques. *New York: Marcel Dekker.*

M. Engelhardt and L.J. Bain (1974). Some results on point estimation for the two-parameter Weibull or extreme-value distribution. *Technometrics 16, 49–56.*

H.J. Khamis (1997). The δ-corrected Kolmogorov-Smirnov test for the two-parameter Weibull distribution. *Journal of Applied Statistics 24, 301-317.*

M.V. Menon (1963). Estimation of the shape and scale parameters of the Weibull distribution. *Technometrics 15, 175–182.*

S.S. Shapiro and C.W. Brain (1987). W-test for the Weibull distribution. *Commun. Statist. -Simula. 16, 209–219.*

G.R. Shorack and J.A. Wellner (1986). Empirical Processes with Applications to Statistics. *New York: Wiley.*

J.S. White (1965). Linear estimation for log-Weibull distribution. *Annals of Reliability and Maintainability 4, 915–925.*

J. Wu, A.C.M. Wong and K.W. Ng (2005). Likelihood-based confidence interval for the ratio of scale parameters of two independent Weibull distributions. *J. Statist. Plann. Inference 135, 487–497.*

Z. Yang, M. Xie and A.C.M. Wong (2007). A unified confidence interval for reliability-related quantities of two-parameter Weibull distribution. *Journal of Statistical Computation and Simulation 77, 365–378.*

Table 1: Critical points at level 0.10 are simulated from Cramer-von Mises test statistic, the test statistics T_2 and $\tilde{T}_2$ for sample size $n = 20, 50, 100, 200, 500, 1000$.

	$CM_{cri,MLE}$	$CM_{cri,MM}$	$\tilde{T}_{cri}$	$T_{cri,MLE}$	$T_{cri,MM}4$
$n = 20$	0.1258	0.1061	0.0072	0.0062	0.0061
$n = 50$	0.1299	0.1043	0.0066	0.0061	0.0060
$n = 100$	0.1324	0.1038	0.0063	0.0060	0.0060
$n = 200$	0.1335	0.1030	0.0061	0.0059	0.0059
$n = 500$	0.1344	0.1029	0.0060	0.0059	0.0059
$n = 1000$	0.1347	0.1027	0.0059	0.0059	0.0059

Table 2: Critical points at level 0.05 are simulated from Cramer-von Mises test statistic, the test statistics T_2 and $\tilde{T}_2$ for sample size $n = 20, 50, 100, 200, 500, 1000$.

	$CM_{cri,MLE}$	$CM_{cri,MM}$	$\tilde{T}_{cri}$	$T_{cri,MLE}$	$T_{cri,MM}4$
$n = 20$	0.1554	0.1280	0.0089	0.0074	0.0073
$n = 50$	0.1613	0.1262	0.0080	0.0072	0.0071
$n = 100$	0.1655	0.1263	0.0075	0.0071	0.0070
$n = 200$	0.1670	0.1252	0.0072	0.0070	0.0070
$n = 500$	0.1683	0.1249	0.0070	0.0070	0.0070
$n = 1000$	0.1686	0.1250	0.0070	0.0070	0.0070

Table 3: Powers for Cramer-von Mises test denoted by CM and the new test statistic T_2 are calculated based on the critical points given in Table 2 with the same sample size.

	CM	T_2	CM	T_2	CM	T_2
	$n = 20$	$n = 20$	$n = 50$	$n = 50$	$n = 100$	$n = 100$
Beta(0.5,0.5)	0.59	0.54	0.98	0.92	1.0	1.0
Beta(2,2)	0.16	0.23	0.41	0.49	0.77	0.81
N(0,1)	0.21	0.21	0.50	0.38	0.81	0.65
TruncN(-1,1)	0.18	0.27	0.51	0.58	0.88	0.90

Table 4: Powers for Cramer-von Mises test with either maximum likelihood estimator or moment estimator ($CM_M M$), test statistic T_2 with either maximum likelihood estimator or momemt estimator are calculated based on the critical points $CM_{cri,MLE}, CM_{cri,MM}$ and $\tilde{T}_{cri}$ given in Table 2 with $n = 1000$.

	CM with MLE	CM with MM	T_2 with MLE	T_2 with MM
$(a, b, \rho) = (-2, 1, 0.1)$	0.10	0.13	0.10	0.09
$(a, b, \rho) = (-1, 1, 0.1)$	0.15	0.12	0.11	0.11
$(a, b, \rho) = (-1, 2, 0.1)$	0.06	0.06	0.06	0.05
$(a, b, \rho) = (-2, 1, 0.3)$	0.64	0.79	0.65	0.65
$(a, b, \rho) = (-1, 1, 0.3)$	0.73	0.62	0.64	0.62
$(a, b, \rho) = (-1, 2, 0.3)$	0.12	0.11	0.13	0.13
$(a, b, \rho) = (-2, 1, 0.5)$	0.99	1.0	1.0	1.0
$(a, b, \rho) = (-1, 1, 0.5)$	0.97	0.95	0.98	0.98
$(a, b, \rho) = (-1, 2, 0.5)$	0.25	0.28	0.38	0.38
$(a, b, \rho) = (-2, 1, 0.7)$	1.0	1.0	1.0	1.0
$(a, b, \rho) = (-1, 1, 0.7)$	1.0	1.0	1.0	1.0
$(a, b, \rho) = (-1, 2, 0.7)$	0.65	0.71	0.82	0.82