Conductors and minimal discriminants of hyperelliptic curves

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What are conductors and minimal discriminants?

Degenerating family of hyperelliptic curves

Measures of degeneracy

Artin conductor

Minimal discriminant

How are these related?

\[ y^2 = (x - 1)(x^2 - t) \]

\[ y^2 = x^3 - t \]
How are conductors and minimal discriminants related?

Earlier results: (small genus, all residue characteristics)

- If \( g = 1 \), then \( \text{Art}^+(X) = \Delta_X \). [Ogg-Saito formula]
- If \( g = 2 \), then Liu showed that \( \text{Art}^+(X) \leq \Delta_X \). He showed that equality does not always hold.
How are conductors and minimal discriminants related?

Earlier results: (small genus, all residue characteristics)

- If $g = 1$, then $\text{Art}^+(X) = \Delta_X$. [Ogg-Saito formula]
- If $g = 2$, then Liu showed that $\text{Art}^+(X) \leq \Delta_X$. He showed that equality does not always hold.

Question: Does $\text{Art}^+(X) \leq \Delta_X$ hold for hyperelliptic curves of arbitrary genus $g$?

Today:

- Yes, if the residue characteristic is $> 2g + 1$. [S.]
  - Combinatorial restrictions for equality when $g \geq 2$.
- Yes, if the residue characteristic is $\neq 2$. [Joint work in progress with Obus]
1 Introduction
2 Definitions
3 Computational tools
4 Proof strategies in examples
**Notation**

\(R\): complete discrete valuation ring

\(K\): fraction field of \(R\)

\(k\): residue field of \(R\), algebraically closed, \(\text{char} \neq 2\)

\(\overline{K}\): a fixed separable closure of \(K\)

\(G_K\): Galois group of \(\overline{K}/K\)

\(\nu\): valuation \(\overline{K} \to \mathbb{Q} \cup \{\infty\}\)

\(t\): a uniformizer of \(R\), i.e., \(\nu(t) = 1\).

**Examples:** \(\C[[t]], \widehat{\Z_{unr}}\)

\(X\): smooth hyperelliptic \(K\)-curve

\(g\): genus of \(X\)
Definition: The minimal discriminant $\Delta_X$ of $X/K$ is the nonnegative integer

$$
\Delta_X := \min_{f(x) \in R[x], y^2 = f(x), \text{eqn. for } X \in R} \nu(\text{disc}(f)).
$$

An example: $K = \mathbb{C}((t))$

$C_1: y^2 = x(x - t)(x - 2t)(x - 3t) \sim \nu(\text{disc}(f)) = 2 \binom{4}{2}.$

$C_2: y'^2 = x'(x' - 1)(x' - 2)(x' - 3) \sim \nu(\text{disc}(f)) = 0.$

Here $C_1 \cong_K C_2$ via $x' = \frac{x}{t}, y' = \frac{y}{t^2} \sim \Delta_X = 0.$
Fix a prime $\ell \neq \text{char } k$. For any curve $C$ over an algebraically closed field of char $\neq \ell$, let

$$\chi(C) := \sum_{i=0}^{2} (-1)^i \dim H^i_{\text{ét}}(C, \mathbb{Q}_\ell).$$

$\delta$: Swan conductor for the $G_K$ representation $H^1(X_K, \mathbb{Q}_\ell)$ (integer, $\geq 0$, measure of wild ramification).

$X^{\min}$: minimal proper regular $R$-model of $X$.

**Definition:** The Artin conductor $\text{Art}^+(X)$ of $X/K$ is

$$\text{Art}^+(X) := \chi(X^{\min}_k) - \chi(X^{\min}_K) + \delta.$$
Properties of the Artin Conductor

- Art\(^+\)(X) is independent of \(\ell\).
- Art\(^+\)(X) \(\geq 0\).
  \[
  \text{Art}^+(X) = 0 \iff X_{\min} \to \text{Spec } R \text{ is smooth or } g = 1 \text{ and } (X_k)_{\text{red}} \text{ is smooth.}
  \]
- Let \(n\) be the number of components of \(X_{\min}^k\) and let \(\epsilon\) be the tame conductor for the \(G_K\) representation \(H^1(X_{\overline{K}}, \mathbb{Q}_\ell)\). Then,
  \[
  \text{Art}^+(X) = (n - 1) + \epsilon + \delta.
  \]
- When \(X_{\min}\) is regular and semi-stable,
  \[
  \text{Art}^+(X) = \# \text{ singular points of } X_k^{\text{min}}.
  \]
Theorem (S.)

Let $K$ be the fraction field of a Henselian discrete valuation ring. Let $X$ be a smooth hyperelliptic curve over $K$ of genus $g \geq 1$. Assume that the residue characteristic is $> 2g + 1$. Then,

$$\text{Art}^+(X) \leq \Delta_X.$$
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Remark: Suffices to find ONE proper regular model $\mathcal{X}'$ such that

$$\text{Art}^+(\mathcal{X}') \leq \Delta_{\mathcal{X}}.$$
Explicit regular models when char \( k \neq 2 \)

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\text{Art}^{+}(X) \leq \text{Art}^{+}(\mathcal{X}) \leq \Delta_X.
\]
Explicit regular models when char $k \neq 2$

**Remark:** Suffices to find ONE proper regular model $\mathcal{X}$ such that

$$\text{Art}^+(X) \leq \text{Art}^+(\mathcal{X}) \leq \Delta_X.$$ 

Two reasons for non regular Weierstrass models:

- Components of $\text{div} \ f \subset \mathbb{P}^1_R$ intersect.
  (Example: $K = \mathbb{C}((t))$, $y^2 = x(x - t)(x - 1)$.)

- Components of $\text{div} \ f \subset \mathbb{P}^1_R$ are not regular curves.
  (Example: $K = \mathbb{C}((t))$, $y^2 = x^3 - t^2$.)
Explicit regular models when char $k \neq 2$

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- Components of $\text{div} \, f \subset \mathbb{P}^1_R$ are not regular curves.
  
  (Example: $K = \mathbb{C}((t))$, $y^2 = x^3 - t^2$.)

**Solution:** Blow-up $\mathbb{P}^1_R$ first **before** taking a double cover.
Lemma

Let $\text{Bl} \mathbb{P}_R^1$ be an arithmetic surface birational to $\mathbb{P}_R^1$. Let $f$ be an element of the function field of $\mathbb{P}_R^1$. Assume that the odd multiplicity components of the divisor of $f$ on $\text{Bl} \mathbb{P}_R^1$ are disjoint and regular.

Then, the normalization of $\text{Bl} \mathbb{P}_R^1$ in $K(x, \sqrt{f(x)})$ is a proper regular model for the hyperelliptic curve given by $y^2 = f(x)$. 
Lemma

Let $\text{Bl } \mathbb{P}^1_R$ be an arithmetic surface birational to $\mathbb{P}^1_R$. Let $f$ be an element of the function field of $\mathbb{P}^1_R$. Assume that the odd multiplicity components of the divisor of $f$ on $\text{Bl } \mathbb{P}^1_R$ are disjoint and regular. Then, the normalization of $\text{Bl } \mathbb{P}^1_R$ in $K(x, \sqrt{f(x)})$ is a proper regular model for the hyperelliptic curve given by $y^2 = f(x)$.

Explicit regular model:
Let $y^2 = f(x)$ be an equation for $X$ with $f(x) \in R[x]$ and $\Delta_X = \Delta_f$. Let $\text{Bl } \mathbb{P}^1_R$ be the (minimal) blowup of $\mathbb{P}^1_R$ satisfying the conditions above and $\mathcal{X}_f$ the associated proper regular model of $X$. 
Computational tools

- Riemann-Hurwitz formula: If $\mathcal{X} \rightarrow \mathcal{Y}$ is a double cover of arithmetic surfaces, branched over the divisor $B$, then,

\[
\text{Art}^+(\mathcal{X}) = [2\chi(\mathcal{Y}_k) - \chi(B_k)] - [2\chi(\mathcal{Y}_K) - \chi(B_K)] + \delta.
\]

- Inclusion-exclusion/additivity for $\chi$ (good for induction!).
Riemann-Hurwitz formula: If $\mathcal{X} \to \mathcal{Y}$ is a double cover of arithmetic surfaces, branched over the divisor $B$, then,

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Inclusion-exclusion/additivity for $\chi$ (good for induction!).

Additional tools:
- $\text{char } k > 2g + 1 \leadsto \delta = 0$
- Roots of $f \leadsto$ Metric tree of $f$
- Induction on the metric tree
- Abhyankar’s Inversion formula

Key inductive inequality:

$$\Delta_f - \Delta_{f_{\text{new}}} = n(n - 1) \geq 2 = \text{Art}^+(\mathcal{X}_f) - \text{Art}^+(\mathcal{X}_{f_{\text{new}}}) \ (\because n \geq 2).$$
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Roots of $f \leadsto$ Metric tree of $f$

$$
t^{2/3} + t^{5/6}, t^{2/3} - t^{5/6},
\omega t^{2/3} - \omega^2 t^{5/6}, \omega t^{2/3} + \omega^2 t^{5/6},
\omega^2 t^{2/3} + \omega t^{5/6}, \omega^2 t^{2/3} - \omega t^{5/6}
$$
Inductive process on metric trees using Abhyankar’s inversion formula

(In the example below, \( a = 2, b = 3 \).) 

\( b \) identical subtrees \( \sim \) \( a \) identical subtrees.

distance \( a/b \) from \( \eta \) \( \sim \) distance \( (b/a) - 1 \) from \( \eta \).

New subtree metric = (Old subtree metric) \( \cdot \) \( b/a \).
Proof in an easy example, $K = \mathbb{C}((t))$

\[ f(x) = x(x - 1 - t)(x - 1 - 2t)(x - 1 - 3t)(x - 1 - 4t) \]

\[ f^{\text{new}}(x) = (x - 1)(x - 2)(x - 3)(x - 4) \]

\[ \text{Art}^+(X_f) - \text{Art}^+(X_{f^{\text{new}}}) = 2. \]

\[ \Delta_f - \Delta_{f^{\text{new}}} = 2\binom{4}{2} = 12. \]

\[ \text{Art}^+(X_{f^{\text{new}}}) = \Delta_{f^{\text{new}}} = 0. \]
Let $K = \mathbb{Q}^\text{unr}_p$, $p$ odd.

$y^2 = x^p - p$

$y^2 = x^p - p^2$
Examples where $\delta \neq 0$

Let $K = \mathbb{Q}_p^{unr}$, $p$ odd.

$y^2 = x^p - p$

- Weierstrass model is regular!
- $\text{Art}^+(X) = [2\chi(\mathcal{V}_k) - 2\chi(\mathcal{V}_{K^s})] - [\chi(B_k) - \chi(B_{K^s})] + \delta = p - 1 + \delta$
- $\delta = \Delta_{K(p^{1/p}/K) = [K(p^{1/p}) : K]} + 1 = \Delta_f - p + 1.$

$y^2 = x^p - p^2$
Let \( K = \hat{Q}_p^{\text{unr}} \), \( p \) odd.

\[
y^2 = x^p - p
\]

- Weierstrass model is regular!
- \( \text{Art}^+(X) = [2\chi(\mathcal{V}_k) - 2\chi(\mathcal{V}_{\overline{K}})] - [\chi(B_k) - \chi(B_{\overline{K}})] + \delta = p - 1 + \delta \)
- \( \delta = \Delta_{K(p^{1/p})/K} - [K(p^{1/p}) : K] + 1 = \Delta_f - p + 1. \)

\[
y^2 = x^p - p^2
\]

- \( \delta = \Delta_{K(p^{2/p})/K} - [K(p^{2/p}) : K] + 1 \)
  \[
  = \Delta_{K(p^{1/p})/K} - [K(p^{1/p}) : K] + 1 \\
  = \Delta_f - 2(\nu_p(p^{2/p}) - \nu_p(p^{1/p})) \left(\frac{p}{2}\right) - p + 1 \\
  = \Delta_f - 2(p - 1). \)
Examples where $\delta \neq 0$

Let $K = \hat{\mathbb{Q}}_{p}^{unr}$, $p$ odd.

$y^2 = x^p - p$

- Weierstrass model is regular!
- $\text{Art}^{+}(X) = [2\chi(\mathcal{Y}_K) - 2\chi(\mathcal{Y}_{K^p})] - [\chi(B_k) - \chi(B_{K^p})] + \delta = p - 1 + \delta$
- $\delta = \Delta_{K(p^{1/p})/K} - [K(p^{1/p}) : K] + 1 = \Delta_f - p + 1.$

$y^2 = x^p - p^2$

- Weierstrass model is not regular! Need $(p - 1)/2$ blowups of $\mathbb{P}^1_R$.

- $\delta = \Delta_{K(p^{2/p})/K} - [K(p^{2/p}) : K] + 1$
  - $= \Delta_{K(p^{1/p})/K} - [K(p^{1/p}) : K] + 1$
  - $= \Delta_f - 2(\nu_p(p^{2/p}) - \nu_p(p^{1/p}))(\frac{p}{2}) - p + 1$
  - $= \Delta_f - 2(p - 1).$
Examples where $\delta \neq 0$

Let $K = \widehat{\mathbb{Q}_p}^{\text{unr}}$, $p$ odd.

$y^2 = x^p - p$
- Weierstrass model is regular!
- $\text{Art}^+(X) = [2\chi(Y_k) - 2\chi(Y_{\overline{K}})] - [\chi(B_k) - \chi(B_{\overline{K}})] + \delta = p - 1 + \delta$
- $\delta = \Delta_K(p^{1/p}/K) - [K(p^{1/p}) : K] + 1 = \Delta_f - p + 1$.

$y^2 = x^p - p^2$
- Weierstrass model is not regular! Need $(p - 1)/2$ blowups of $\mathbb{P}^1_R$.
- $\text{Art}^+(X) = 2(p - 1) + \delta$
- $\delta = \Delta_K(p^{2/p}/K) - [K(p^{2/p}) : K] + 1$
- $\delta = \Delta_K(p^{1/p}/K) - [K(p^{1/p}) : K] + 1$
- $\delta = \Delta_f - 2(\nu_p(p^{2/p}) - \nu_p(p^{1/p}))\left(\frac{p}{2}\right) - p + 1$
- $\delta = \Delta_f - 2(p - 1)$. 
Finally . . .

Thank you!
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Explicit Methods in Arithmetic Geometry in Characteristic p

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Application deadline: February 15