CS 3510 - Spring 2009
Practice Problems 1 Solutions

1. Let a, b, and c be positive and real numbers. Show that $a^{\log_b c} = c^{\log_b a}$.

   Proof. First note that

   $$\begin{aligned}\log_b c &= \log_b c / \log_b a * \log_b a \\ &= \log_a c * \log_b a.\end{aligned}$$

   Then we have

   $$\begin{aligned}a^{\log_b c} &= a^{\log_a c * \log_b a} \\ &= (a^{\log_a c})^{\log_b a} \\ &= c^{\log_b a}.\end{aligned}$$

   □

2. Let b be a real number greater than 1, and let x and y be positive real numbers. Show that $\log_b(x^y) = y \log_b x$.

   Proof. Let $z = \log_b x$. By definition, this means that $b^z = x$. Therefore $b^{zy} = x^y$. Taking the logarithm of both sides, we find that

   $$\log_b b^{zy} = \log_b x^y,$$

   So

   $$zy = \log_b x^y.$$

   Substituting back for z gives the desired claim. □

3. Let a and b be real numbers greater than 1, and let x be a positive real number. Show $\log_a x = \log_b x / \log_b a$.

Proof. Let $\log_a x = u$, so $x = a^u$). Also, let $\log_b x = v$, so we have $x = b^v$ and let $\log_b a = w$, so $a = b^w$. It follows that

$$\begin{aligned} x &= a^u = b^v \\ \Rightarrow (b^w)^u &= b^v \\ \Rightarrow b^{wu} &= b^v. \end{aligned}$$

But exponentiation is one-to-one, so it follows that $wu = v$ and therefore $\log_a x = \log_b x / \log_b a$. □

4. Let m be a positive integer. Show that $a \equiv b \pmod{m}$ if $a \pmod{m} = b \pmod{m}$.

   Proof. If $a \pmod{m} = b \pmod{m}$, then a and b have the same reminder when divided by m. Hence $a = q_1 m + r$ and $b = q_2 m + r$, where $0 \leq r < m$. It follows that $a - b = (q_1 - q_2)m$ so that $m|(a - b)$. It follows that $a \equiv b \pmod{m}$. □

5. Let m be a positive integer. Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$

   Proof. Since $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, there are integers s, and t with $b = a + sm$ and $d = c + tm$. Hence,

   $$b + d = (a + sm) + (c + tm) = (a + c) + m(s + t)$$

   and

   $$bd = (a + sm)(c + tm) = ac + m(at + cs + stm).$$

   Hence,

   $$a + c \equiv b + d \pmod{m}$$

   and

   $$ac \equiv bd \pmod{m}$$

   □

6. Find $2^{1744} \pmod{127}$.

   Notice that $2^7 = 128 = 1 \pmod{127}$.
   Then,

   $$\begin{aligned} 2^{1744} \pmod{127} &\equiv 2^{7*249} * 2 \pmod{127} \\ &\equiv (2^7 \bmod 127)^{249} * 2 \pmod{127} \\ &\equiv 1^{249} * 2 \pmod{127} \\ &\equiv 2 \pmod{127}. \end{aligned}$$

7. Find the unit's digit of 287^{3503}.

First notice that $287^{3503} \pmod{10}$ is the unit's digit, and this is equivalent to $(287 \bmod 10)^{3503} \equiv 7^{3503} \pmod{10}$.

If we look at successive powers of 7 mod10, we find $7^0 = 1 \pmod{10}$, $7^1 \equiv 7 \pmod{10}$, $7^2 \equiv 9 \pmod{10}$, $7^3 \equiv 7^2 * 7 \equiv 9 * 7 \equiv 3 \pmod{10}$, and then $7^4 \equiv 7^3 * 7 \equiv 3 * 7 \equiv 1 \pmod{10}$. At this point the sequence $\{1, 7, 9, 3\}$ just repeats for successive powers of 7, so $7^{4k} \equiv 1 \pmod{10}$ for every integer k. Therefore,

$$\begin{aligned} 287^{3503} \bmod 10 &\equiv 7^{3503} \pmod{10} \\ &\equiv (7^{4*875} * 7^3) \pmod{10} \\ &\equiv 1^{875} * 7^3 \pmod{10} \\ &\equiv 3 \pmod{10}. \end{aligned}$$

8. What is $3^{602} \pmod 7$? (Hint: Use Fermat's little theorem.)

Fermat's little theorem tells us

$$3^6 \equiv 1 \pmod 7.$$

This tells us that

$$\begin{aligned} 3^{602} &\equiv 3^{6*100+2} \pmod 7 \\ &\equiv (3^6 \bmod 7)^{100} * 3^2 \pmod 7 \\ &\equiv 1^{100} * 3^2 \pmod 7 \\ &\equiv 2 \pmod 7. \end{aligned}$$