Quiz 4
CS3510 (and 3511), Algorithms

1. Consider the problem of making change with a set of unlimited number of coins of denominations $X[1], ..., X[n]$. Suppose you want to make change for an amount A by using the minimum number of coins.

   (a) No solution: Let's say we have denominations 25 and 10. If we want to make change for 40 using the greedy approach, we would select 25 and then 10. This gives us 35, with no denominations remaining. However, we see that it is possible to make change for 40 using four coins of value 10.

   Suboptimal: Let's say we have denominations of 25, 10, and 1. Using the same example as above, if we want to make change for 40, the greedy approach yields 25+10+1+1+1+1+1 = 40. However, 10+10+10+10, makes change for 40 with fewer coins.

   (b) This problem is essentially knapsack with repetitions, but we are instead attempting to minimize the number of coins. N(v) = minimum number of coins for value v.

   $$N[v] = min\{N[v - x_i] + 1 : x_i \leq v\}$$

   The initial condition is $N[0] = 0$. This basically means that need zero coins to achieve value 0.

   (c) $N[0] = 0$
   for $i = 1$ to A:
   $N[v] = min\{N[v - x_i] + 1 : x_i \leq v\}$
   return $N[A]$

   (d) The running time for this algorithm is $O(nA)$.

2. Answer true or false to each part and briefly explain your answer.

   (a) True. Any solution to problem A can be verified in polynomial time. B is at most as difficult as A and so it can also be verified in polynomial time, placing it in NP.

   (b) True. T is a more general version of problem S. Since S is NP-hard, we know that T must be at least equally difficult and NP-hard as well.

   (c) False. T is a specialized version of problem S. There is no guarantee that it is NP-hard, for an example, see 2-SAT.

   (d) True. By definition, all NP-Complete problems can be reduced to in polynomial time.

   (e) False. This would mean that C provides a polynomial time solution for all NP complete problems. Such a reduction has not been found and is generally believed not to exist.

3. **CLIQUE-AND-INDEPENDENT-SET**

   (a) Given a graph G and an integer k, a clique is a set of k nodes that are fully connected.

   (b) This proof is not sufficient. If you are given a graph that does not have a clique, but has an independent set, it will yield an instance of CAIS. This is only supposed to occur when the original graph has a clique, so this reduction is invalid.

   (c) This reduction is sufficient. The resulting graph will always have an independent set of size k, but it will only have a clique and an independent set, when the original graph has a clique.

4. **HALF-CLIQUE**

   (a) HALF-CLIQUE is in **NP** because given a graph G and a set of $n/2$ nodes, we can verify that these vertices all have edges to each other in polynomial time. For each node we look at its adjacency list and verify that the other $n/2 - 1$ nodes are in it. This is $O(n^2)$.

(b) Reduction from CLIQUE to HALF-CLIQUE.

(c) Reduction and justification:
Given an input G,k to CLIQUE, we want to produce a HALF-CLIQUE iff the original graph has a CLIQUE. If k is equal to $n/2$, then the original graph can be passed to HALF-CLIQUE unmodified. The remaining two cases to handle is if k in the original graph is greater than $n/2$ or less than $n/2$.

If $k > n/2$, then we need to add nodes to the graph without changing the clique size. This is done by adding disconnected vertices. Once the graph has $2k$ nodes, then this new graph will have a half-clique only when the original graph has a k-clique.

If $k < n/2$, we can either increase the size of cliques in the original graph or decrease the total number of nodes without affecting any k-cliques that may be present. We shall do the former. We can add vertices to the graph that have edges to all other nodes currently present in the graph. For each vertex we add, all cliques will increase in size by 1. The question remains of how many total nodes should we add to the original graph. Let a be the number of nodes that we add, we want $k + a$ to equal $(n + a)/2$, which means that we should add $a = n - 2k$ nodes. This means that only cliques of size k will grow large enough to be half-cliques.