An Introduction to Sampling and Counting

Introduction

We begin by discussing perfect matchings on a graph. A graph $G$ is defined by its vertex set and edge set, denoted by $G = (V_1, V_2, E)$. A bipartite graph is a graph which has the property $E \subseteq V_1 \times V_2$. For now, we restrict our attention to bipartite graphs with $|V_1| = |V_2| = n$. A perfect matching is a set $M \subseteq E$ s.t. $\forall u \in V_1 \ \exists! \ v \in V_2$ s.t. $(u, v) \in M$. In Figure 1, bold lines indicate edges which form a perfect matching.

Computational Problems

- Input is given as an adjacency matrix $A$

  $$V_1 = \{1, \ldots, n\}$$
  $$V_2 = \{1, \ldots, n\}$$

  $$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

1) Decision Problem
   Input: $G = (V_1, V_2, E)$
   Output: Does $G$ contain a perfect matching?

2) Search Problem
   Input: $G = (V_1, V_2, E)$
   Output: A perfect matching (or it indicates that none exists).

3) Counting
   Input: $G = (V_1, V_2, E)$
   Output: The number of perfect matchings (written in binary).
4) Sampling
Input: $G = (V_1, V_2, E)$
Output: A perfect matching chosen uniformly from the set of all perfect matchings (or message saying none exists).

**How many matchings are there?**

**Example 1:** Complete Graph - denoted by $K(n,n)$.
Number of perfect matchings = $n!$

**Example 2:** Lattice Graph - $R \subseteq \mathbb{Z}^2$

We can consider the equivalent problem of domino tilings on lattice regions of size $n = 2r \times 2r$ (in the dual graph), see Figure 2. By considering blocks of four cells we obtain the lower bound $2r^2 = \frac{2n}{4}$. We can also consider the “arrow” representation to obtain an upper bound of $4^n$. So

$$2^{n/4} \leq C_n \leq 4^n$$

where $C_n$ is the number of tilings.

**Applications**

1) Statistical Mechanics: the dimer problem
   Invented - 1937 (Fowler and Rushbrooke)
   Progress - 1961 (Fisher, Kastelyn and Temperley)
   
   a) Approximate Counting
      “Estimate” number of solutions
   b) Approximate Sampling
      Output a matching “close” to the uniform distribution
2) Complexity Theory

\[ \text{Det} = \sum_{\pi} \left( \text{sgn}(\pi) \prod_i A_{i\pi(i)} \right) \]

The \( \text{Det} \) is in \( P \), i.e. \( \exists \) a polynomial time algorithm to determine it.

\[ \text{Perm} = \sum_{\pi} \left( \prod_i A_{i\pi(i)} \right) \]

The \( \text{Perm} \) is in \#P-complete.

\textbf{Fact} : the number of perfect matchings = \( \text{Perm}(A) \).