Math 8213A – Rapidly Mixing Markov Chains Lecturer: *Dana Randall*

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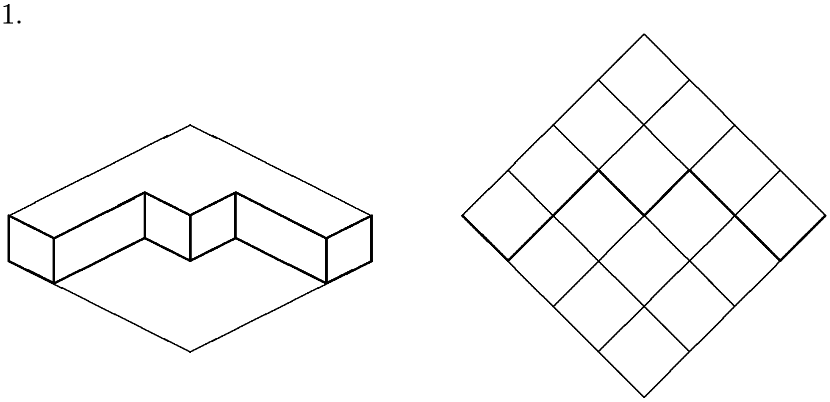
The 1-path case for lozenge tilings

For simplicity, we will consider generating lozenge tilings of a $(m, m, 1)$ hexagonal region (see Figure 1), this corresponds to only one path in the associated routing. The two Markov Chains defined last time are identical in the 1-path case, since all towers have height 1 in this case. It is convenient to think that the Markov Chain is applied to the associated path, the algorithm phrased in this way is the following:

1. Pick a random vertical line and flip an unbiased coin c.
   If $c = \textit{heads}$ and the path has a valley at the intersection with the vertical line, rotate up.
   If $c = \textit{tails}$ and the path has a peak at the intersection with the vertical line, rotate down.
2. Repeat Step 1.



(a) A lozenge tiling (a) The corresponding path

Figure 1: One-path case for lozenge tilings

We will run two coupled copies of this algorithm, one starts from the highest possible path, the other starts from the lowest one. They are coupled in the sense that they use the same vertical line and the same coin toss in any iteration. Since $P_1 \succeq P_2$ at the beginning of the algorithm, P_1 will remain above P_2 at any time $t \geq 0$. Denote by $\Phi_t(P_1, P_2)$ the area between paths P_1 and P_2 at time t.

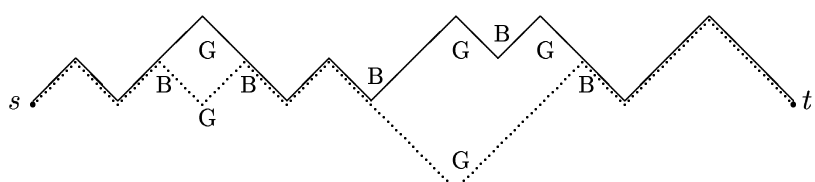


Figure 2: Proof of lemma 6

Define a "good" point to be a peak on P_1 or a valley on P_2, similarly, "bad" points are the valleys of P_1 and the peaks of P_2. (These names suggest that a vertical line passing through a good point might decrease Φ_t, while one passing through a bad point might increase it, see Figure 2.) The expected change in Φ_t is then given by:

$$E\left[\Delta\Phi_t\right] = \frac{1}{cn}(\#\text{Bad points} - \#\text{Good points}),$$

where c is a constant.

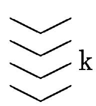
We claim that, for any t, the number of good points is greater or equal with the number of bad points; this implies that $E[\Delta\Phi_t] \leq 0$. Indeed, consider a "bubble" of the region between P_1 and P_2, and let A and B be the two points on the border of the bubble that are in $P_1 \cap P_2$. On P_1, there is one more good point between A and B, and the same is true for P_2. If A (or B) is counted twice as bad point (on both P_1 and P_2), it must be the case that it is also appearing on the border of another bubble, so, overall, the number of good points is at least as large as the number of bad points.

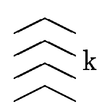
Clearly, $\Phi_t \leq \Phi_0$, and, once the two paths overlap, they stay together (i.e. $\Phi_t = 0$ for every $t \geq t_0$ if $\Phi_{t_0} = 0$). As a simplified model, we can think of Φ_t as performing a random walk between Φ_0 and 0, with an absorbing barrier at 0. Intuitively, since $E[\Delta\Phi_t] \leq 0$, the walk is either unbiased, or biased toward 0, so the time to get to 0 should be $O(\Phi_0^2)$. We will give the proof later, in the case of multiple paths.

Multiple paths

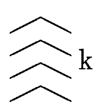
All the discussion above applies to multiple paths as well. In this case, we still have a distributive lattice structure on the set of routings according with the "above" relation. The Markov Chain acting on routings is:

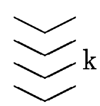
1. Start with an arbitrary lozenge tiling
2. Pick a hexagonal window uniformly at random and $r \in_u (0,1)$.

   If the window looks like 

 , change it to 

 if $r \leq 1/2k$.

   If the window looks like 

 , change it to 

 if $r \geq 1 - 1/2k$.

3. Repeat Step 2.

Define $\Phi_t(R_1, R_2)$ as the sum of areas between the corresponding pairs of paths. Then the following properties are true (the proof is similar to that for the one-path case):

1) $0 \leq \Phi_t \leq B$ for some $B > 0$;

2) $E[\Delta\Phi_t] \leq 0$;

3) $\Phi_t = 0$ implies $\Phi_{t+h} = 0$ for all $h \geq 0$.

Moreover, there is at least one point where we can make progress, which implies:

4) If $\Phi_t > 0$, then $E[(\Delta\Phi_t)^2] > V$, for some constant V $(V = O(\frac{1}{n}))$.

Let $T^{x,y}$ be the time needed for coupling if we start from states x and y.

Claim : $E[T^{x,y}] \leq \frac{\Phi_0(2B-\Phi_0)}{V}$.

We will use the Optional Stopping Theorem:

Theorem : Let Y_t be a random variable bounded by above, with bounded variance, and which satisfies the sub-martingale condition (i.e. $E\left[Y_{t+1}|Y_t,...,Y_0\right] \geq Y_t$). If $\widehat{T}$ is a stopping event such that $E\widehat{[T]} < \infty$, then $E[Y_{\widehat{T}}] \geq E[Y_0]$.

Proof of the claim: Define the stochastic process $Z(t) = \Phi_t^2 - 2B\Phi_t - Vt$. Then $Z(t)$ is a sub-martingale:

$$\begin{gathered} E\left[Z(t+1)|Z(t),...,Z(0)\right] - Z(t) = \\ (\Phi_t + E\left[\Delta\Phi_t\right])^2 - 2B(\Phi_t + E\left[\Delta\Phi_t\right]) - V(t+1)) - (\Phi_t^2 - 2B\Phi_t - Vt) = \\ 2\left(\Phi_t - B\right) E\left[\Delta\Phi_t\right] + \left(E\left[\Delta\Phi_t\right] - V\right) \geq 0 \end{gathered}$$

Applying the Optional Stopping Theorem for $\widehat{T} = T^{x,y}$ we get that

$$E\left[Z(T^{x,y})\right] \geq E\left[Z(0)\right] = Z(0).$$

Since $\Phi_{T^{x,y}} = 0$, $E\left[Z(T^{x,y})\right] = -VT^{x,y}$, hence

$$-VT^{x,y} \geq Z(0) = \Phi_0^2 - 2B\Phi_0.$$

By applying the claim to the highest/lowest routings as starting routings, we obtain an upper bound on the expected coupling time. In this case, $B = \Phi_0 = O(n^{1.5})$, therefore we get a running time of $O(n^4)$.