Today we introduce some basics of Percolation Theory and the FKG Inequality.

1 Percolation

Let $\Lambda = \mathbb{Z}^d$ denote the integer lattice and $B$ denote the set of bonds (edges) in the lattice. Let $\Omega = \{0,1\}^B = \prod_{b \in B} \{0,1\}$. For $\omega \in \Omega$, $\omega_b = 1$ means that the bond $b$ is open (occupied), and $\omega_b = 0$ means that bond $b$ is closed (unoccupied). Also, for $\omega \in \Omega$, we let $S(\omega) = \{b : \omega_b = 1\}$ be the collection of bonds which are open. For $x$ a vertex of $\Lambda$, we let $c(x)$ denote the set of vertices connected to $x$ using edges of $S(\omega)$. (I.e. $c(x)$ is the connected component containing $x$.) We write $x \leftrightarrow y$ if $c(x) = c(y)$ (meaning that $x$ and $y$ are in the same connected component), and $x \leftrightarrow \infty$ if $|c(x)| = \infty$.

**Definition 1** Let $0 \leq p \leq 1$ and for any bond $b$ let $\mu_b(1) = p$, $\mu_b(0) = q = 1 - p$. We define the percolation measure on a configuration, $\omega$, to be

$$P_p(\omega) = \prod_{b \in B} \mu_b(\omega_b).$$

The primary questions that we are interested in are when is it likely that $x \leftrightarrow y$, and, more importantly, when is it likely that $0 \leftrightarrow \infty$?

**Definition 2** Let $P_\infty(p) = P_p(0 \leftrightarrow \infty)$. If $P_\infty(p) > 0$ we say that we have percolation.

**Remarks:**

1. $P_\infty(0) = 0$, so there is no percolation.

2. $P_\infty(1) = 1$, so there is percolation.

3. Intuitively, if $P_\infty(p) > 0$, then $P_\infty(p') > 0$ for $p' > p$.

**Definition 3** In $\mathbb{Z}^d$, let $P_c(d) = \inf\{p : P_\infty(p) > 0\}$.

A sketch of $P_\infty(p)$ versus $p$ is given below.
We will prove that this “critical” probability is strictly between zero and one. First, some more definitions that we will use to do this.

Let \( \sigma_d(N) \) be the number of self avoiding walks of length \( N \) in \( \mathbb{Z}^d \).

Let
\[
\lambda(d) = \lim_{N \to \infty} \sigma_d(N)^{\frac{1}{N}}.
\]

Note that we have \( d \leq \lambda(d) \leq 2d - 1 \).

To see the lower bound, start at the origin, and construct a self avoiding walk by choosing one of the \( d \) coordinates and adding 1 to that coordinate (i.e. always walk in the positive direction). Do this for \( N \) steps. The walk constructed in this manner is self avoiding since each coordinate position is a non-decreasing function of \( N \) (and one coordinate increases at each step). The number of walks we can construct this way is \( d^N \), hence \( \sigma_d(N) \geq d^N \).

For the upper bound, start at the origin, choose one of the \( 2d \) vertices connected to the origin, and move there. For each step after that, choose one of the \( 2d - 1 \) other vertices joined to the current one, and move there (i.e. don’t return to the one you just came from). The number of walks constructed in this manner is \( 2d(2d - 1)^{N-1} \), so \( \sigma_d(N) \leq 2d(2d - 1)^{N-1} \).

**Theorem 1** For \( d \geq 2 \) we have \( 0 < P_c(d) < 1 \). More specifically, \( \frac{1}{\lambda(d)} \leq P_c(d) \leq 1 - \frac{1}{\lambda(d)} \).

**Proof:** \( (P_c(d) \geq \frac{1}{\lambda(d)}) \)

The probability that any self avoiding walk of length \( N \) consists of all open bonds is \( p^N \). For any configuration let \( \tau_d(N) \) be the number of self avoiding walks of length \( N \) which are open. Then \( E_p(\tau_d(N)) = p^N \sigma_d(N) \). So

\[
P_\infty(p) \leq P_p(\tau_d(N) \geq 1) \quad \text{ (for all } N \geq 1) \leq E_p(\tau_d(N)) = p^N \sigma_d(N)
\]

Note that \( p < \frac{1}{d} \) suffices for the limit to be 0. In fact, as \( N \to \infty \), we have that \( P_\infty(p) \to 0 \). Therefore, \( P_\infty(p) \geq \frac{1}{\lambda(d)} \).

\( (P_c(d) \leq 1 - \frac{1}{\lambda(d)}) \)

First observe that showing this statement in 2-d is sufficient as \( P_c(d) \leq P_c(2) \) for \( d > 2 \) (since if we have percolation on some 2-d sub-lattice containing the origin, then we have percolation in the \( d \) dimensional lattice as well). Therefore, consider the restriction to two dimensions.

For any configuration \( \omega \), if we don’t have percolation, then there is a simply connected circuit in the dual lattice enclosing the component that contains the origin, where all of the bonds crossing the circuit are closed. Let \( \Gamma_N \) be the number of simply connected circuits of length \( N \) enclosing the origin. Then \( \Gamma_N \leq N \cdot \sigma_d(N - 1) \). (Why? Any circuit of length \( N \) must cross the \( x \)-axis between \((0,0)\) and \((N,0)\). Then starting at \((x + \frac{1}{2}, \frac{1}{2})\) in the dual lattice, the next \( N - 1 \) edges of the circuit must be self avoiding.) So

\[
\sum_{\gamma: \text{circuit}} P_p(\gamma \text{ closed}) \leq \sum_{N=1}^{\infty} q^N N \sigma_d(N - 1).
\]

(Here we note that if \( \lambda(2) \cdot q < 1 \) then the probability of any closed circuit is finite. In fact, as \( q \to 0 \), \( P_p(\text{closed circuit}) \to 0 \).)
Let $F_N$ be the event of no closed circuit of length $\leq N$, and let $G_N$ be the event of no closed circuit of length $\geq N$.

Then
\[
P_\infty(p) \geq P_p(F_N \cap G_N) \\
= P_p(F_N|G_N)P_p(G_N) \\
\geq P_p(F_N)P_p(G_N)
\]

where the last inequality follows from the FKG inequality, which we prove below. If $q < \frac{1}{\chi(2)}$, there's some value of $N$ such that the probability of a closed circuit of length at least $N$ is at most $\frac{1}{2}$, i.e. $P_p(G_N^C) < \frac{1}{2}$ for some $N$. But for any finite $N$ we have $P_p(F_N) > 0$, hence $P_\infty(p) > 0$ (if $q < \frac{1}{\chi(2)}$). \[\square\]

## 2 The FKG Inequality

We will prove the FKG inequality on finite state spaces, although it holds for general percolation spaces.

Consider the state space $\Omega = \mathbb{Z}_n^d$, and $A \subset \Omega$ is an “event.”

**Definition 4** $A$ is an increasing event if for all $\omega \in A$, if $\omega' > \omega$ then $\omega' \in A$.

**Examples**

1. $A = \{ \omega : 0 \leftrightarrow \infty \}$ on the infinite lattice $\mathbb{Z}^d$.
2. $A = \{ \omega : \omega$ has a left-right crossing $\}$ in $\mathbb{Z}_n^2$.

The FKG inequality was discovered by Harris, and by Fortuin, Kasteleyn, and Ginibre.

**Theorem 2 (FKG)** If $A$ and $B$ are increasing events, then
\[
P_p(A \cap B) \geq P_p(A)P_p(B).
\]

We will prove a stronger version of this theorem. First we need a definition.

**Definition 5** Let $\mu$ be a probability measure on $\Omega$. We say that $\mu$ satisfies the FKG condition if
\[
\mu(a \cup b)\mu(a \cap b) \geq \mu(a)\mu(b)
\]
for all $a, b \in \Omega$.

We will prove a stronger version of the FKG inequality due to Holley.

**Theorem 3 (FKG, Stronger Version)** Let $\mu$ be a probability measure on $\Omega$ that satisfies the FKG condition (1), and let $f$ and $g$ be increasing functions on $\Omega$. Then
\[
\sum_{a \in \Omega} f(a)g(a)\mu(a) \geq \sum_{a \in \Omega} f(a)\mu(a) \sum_{b \in \Omega} g(b)\mu(b).
\]
Remarks:

1. $P_p$ is an FKG measure (i.e. it satisfies the FKG condition (1), with equality)

$$\frac{P_p(a \cup b)}{P_p(a)} = \left( \frac{p}{1-p} \right)^{|b\setminus a|} = \frac{P_p(b)}{P_p(a \cap b)}$$

where $b\setminus a$ are the bonds which are open in $b$ but not in $a$.

2. Let $f = 1_A$, i.e. the indicator for the set $A$, and $g = 1_B$. Then, with these $f$ and $g$, the stronger version (Thm, 3) implies the first FKG inequality (Thm, 2).

In the next lecture we will see the proof of the FKG inequality.