Percolation in $\mathbb{Z}^2$

Recall that the critical probability of bond percolation on the $d$-dimensional cartesian lattice is defined by

$$p_c(\mathbb{Z}^d) = \inf\{p : P_p[0 \leftrightarrow \infty] > 0\},$$

where $P_p[0 \leftrightarrow \infty]$ is the probability that the origin is in an infinite open cluster. We have seen that

$$0 < p_c(\mathbb{Z}^d) < 1$$

for any $d \geq 2$. For $d = 2$ we have $p_c(\mathbb{Z}^2) = \frac{1}{2}$. This means that we do not have percolation in $\mathbb{Z}^2$ for $p < \frac{1}{2}$, and we have percolation for $p > \frac{1}{2}$. In this lecture we study what happens at the critical probability.

One of the key ingredients in the study of bond percolation in $\mathbb{Z}^2$ is self-duality. Let $LR(l, k)$ denote the event that there is a left-to-right path of open bonds in a fixed $l \times k$ region of $\mathbb{Z}^2$. A first application of self-duality is:

**Lemma 1** $P_{\frac{1}{2}}[LR(l, l - 1)] = \frac{1}{2}$.

**Proof**: We either have a left-to-right path of open bonds in the box itself, or a top-to-bottom path of closed bonds in the dual box (which is of size $(l - 1) \times l$). Since for $p = \frac{1}{2}$ these two complementary events have equal probability, the claim follows. $\square$

For even $l$, let $B(l)$ denote the $l \times l$ region of $\mathbb{Z}^2$ centered at the origin. Showing that percolation does not happen for $p = \frac{1}{2}$ is equivalent to showing that, with probability 1, there exists a closed circuit around the origin. Our strategy will be to show that, with probability uniformly bounded away from zero, there exists a closed circuit in any annulus of the form $A(l) = B(3l) - B(l)$. More precisely, we will prove:

**Theorem 2** (Russo–Seymour–Welsh) If $\tau = P_{p}[LR(l, l)]$ then

$$P_p[A(l)] \text{ contains an open circuit} \geq (\tau(1 - \sqrt{1 - \tau})^4)^{12}$$

For $p = \frac{1}{2}$ we have $\tau \geq \frac{1}{4}$ (by Lemma 1, $P_{\frac{1}{2}}[LR(l, l - 1)] = \frac{1}{2}$, and the bond that extends any left-to-right path in the $l \times (l - 1)$ box is open with probability $\frac{1}{2}$). Thus, since a circuit has the same probability of being open or closed when $p = \frac{1}{2}$, the RSW theorem gives:

$$P_{\frac{1}{2}}[A(l)] \text{ contains a closed circuit} \geq \frac{1}{4^{12}} \left(1 - \sqrt{\frac{3}{2}}\right)^{48}.$$ 

This implies that for $p = \frac{1}{2}$ we have a closed circuit in at least one of the disjoint annulue $A(l)$, $l = 2, 2 \cdot 3, 2 \cdot 3^2, \ldots$, with probability 1. Hence, we do not have percolation on $\mathbb{Z}^2$ for $p = \frac{1}{2}$.

The key step in the proof of the RSW theorem is:

**Lemma 3** $P_{p}[LR\left(\frac{3}{2}, l\right)] \geq (1 - \sqrt{1 - \tau})^3$
Proof of the RSW theorem: Let
\[
\begin{align*}
\tau_1 &= P_p[LR(\frac{3}{2}l, l)] \\
\tau_2 &= P_p[LR(2l, l)] \\
\tau_3 &= P_p[LR(3l, l)] \\
\tau_4 &= P_p[A(l) \text{ contains an open circuit}]
\end{align*}
\]
We can successively lower bound \(\tau_i\), \(i = 2, 3, 4\), in terms of \(\tau\) and \(\tau_1\) as follows.

First,
\[
\tau_2 \geq \tau_1^2 \tau. \tag{1}
\]
Indeed, a \(2l \times l\) box can be decomposed into two \(\frac{3}{2}l \times l\) boxes that have an overlap of size \(l \times l\):

The \(2l \times l\) box certainly has a left-to-right path of open bonds if both \(\frac{3}{2}l \times l\) boxes have left-to-right open paths and their overlap has a top-to-bottom open path. Thus, inequality (1) follows from the FKG inequality for increasing events.

In a similar way
\[
\tau_3 \geq \tau_2^2 \tau \tag{2}
\]
this time we decompose a \(3l \times l\) box into two \(2l \times l\) boxes with an \(l \times l\) overlap:
Finally,
\[ \tau_4 \geq \tau_3^4, \]
(3)
since \( A(l) \) will certainly contain an open circuit if the two \( 3l \times l \) and the two \( l \times 3l \) boxes that cover it contain left-to-right (resp. top-to-bottom) paths of open bonds:

Combining (1–3) we get
\[ \tau_4 \geq \tau_3^4 \geq (\tau \tau_2)^4 \geq (\tau \tau_1^2)^2 = (\tau_1^3)^4 \]
and the theorem follows from Lemma 3.

To prove Lemma 3 we will need:

**Lemma 4** (*The square root trick.*) If \( A_1 \) and \( A_2 \) are increasing events with equal probability then
\[ P_p[A_1] \geq 1 - \sqrt{1 - P_p[A_1 \cup A_2]} . \]

**Proof:**
\[ 1 - P_p[A_1 \cup A_2] = P_p[A_1 \cap \overline{A_2}] \geq P_p[A_1] \cdot P_p[\overline{A_2}] = (1 - P_p[A_1])^2 . \]

**Proof of Lemma 3:** Without loss of generality, we may assume that the \( 3l \times l \) box is positioned such that its upper-right corner is at \((l, l)\). Let \( B \) be the box \([-\frac{l}{2}, \frac{l}{2}] \times [-\frac{l}{2}, \frac{l}{2}]\), and \( B' \) the box \([0, l] \times [-\frac{l}{2}, \frac{l}{2}]\).

We need to introduce some notations. Let \( T \) be the set of left-to-right paths in \( B \). If \( \pi \in T \), then

- \( y_\pi \) is the point where \( \pi \) crosses for the last time the \( y \) axis;
- \( \pi_r \) is the part of \( \pi \) from \( y_\pi \) to the right border of \( B \);
- \( \pi'_r \) is the reflection of \( \pi_r \) around this border (i.e., around the line \( x = \frac{l}{2} \));
- \( T^- \) (\( T^+ \)) is the set of paths \( \pi \in T \) that have \( y_\pi \leq 0 \) (resp. \( y_\pi \geq 0 \));
- \( L^- \) (\( L^+ \)) is the event that a path in \( T^- \) (resp. in \( T^+ \)) is open;
- \( A_\pi \) is the event that path \( \pi \) is open;
- \( L_\pi \) is the event that path \( \pi \) is the “lowest” open path of \( T \);
$M_\pi^-$ ($M_\pi^+$) is the event that there is an open path from the top of $B'$ to $\pi_r$ (resp. $\pi'_r$);

$N^+$ ($N^-$) is the event that there exists a left-to-right open path in $B'$ starting above (resp. below) the $x$-axis.

Since a configuration in

$$N^+ \cap \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi)$$

guarantees the existence of a left-to-right path of open bonds in $B \cup B'$, we have

$$P_p \left[ LR(\frac{3}{2}l, l) \right] \geq P_p \left[ N^+ \cap \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi) \right].$$

As both $N^+$ and the union event are increasing, by the FKG inequality we get

$$P_p \left[ N^+ \cap \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi) \right] \geq P_p[N^+] \cdot P_p \left[ \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi) \right]$$

Note that $P_p[N^-] = P_p[N^+]$, so, by the square root trick,

$$P_p[N^+] \geq 1 - \sqrt{1 - P_p[N^- \cup N^+]} = 1 - \sqrt{1 - \tau}.$$

Hence, to complete the proof it suffices to show that

$$P_p \left[ \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi) \right] \geq (1 - \sqrt{1 - \tau})^2.$$

Now,

$$P_p \left[ \bigcup_{\pi \in T^-} (M_\pi^- \cap A_\pi) \right] \geq P_p \left[ \bigcup_{\pi \in T^-} (M_\pi^- \cap L_\pi) \right] = \sum_{\pi \in T^-} P_p[M_\pi^- \cap L_\pi] \cdot P_p[L_\pi].$$

Claim: For every $\pi \in T^-$, $P_p[M_\pi^- \cap L_\pi] \geq P_p[M_\pi^-]$.
Let $J_\pi$ denote the set of bonds of $B'$ “trapped” between the left border of $B'$ and $\pi$. Note that $M^-_\pi$ does not depend on whether or not the edges in $J_\pi$ are open. Moreover, note that $M^-_\pi$ and $J_\pi$ are increasing events (this is not true about $L_\pi$). So,

$$P_p[M^-_\pi | L_\pi] = P_p[M^-_\pi | J_\pi] = \frac{P_p[M^-_\pi \cap J_\pi]}{P_p[J_\pi]} \geq \frac{\mathbb{F} \mathbb{K} \mathbb{G} P_p[M^-_\pi] P_p[J_\pi]}{P_p[J_\pi]} = P_p[M^-_\pi],$$

proving the claim.

For every $\pi \in T^-$, $P_p[M^-_\pi] = P_p[M^+_\pi]$ because $\pi'_r$ is the reflection of $\pi_r$. So, using the square root trick and the fact that $P_p[M^-_\pi \cup M^+_\pi] \geq \tau$, we get

$$P_p[M^-_\pi \cup M^+_\pi] \geq 1 - \sqrt{1 - P_p[M^-_\pi \cup M^+_\pi]} \geq 1 - \sqrt{1 - \tau}.$$

Therefore,

$$\sum_{\pi \in T^-} P_p[M^-_\pi | L_\pi] \cdot P_p[L_\pi] \geq (1 - \sqrt{1 - \tau}) \cdot \left( \sum_{\pi \in T^-} P_p[L_\pi] \right) \geq (1 - \sqrt{1 - \tau}) \cdot P_p[T^-].$$

The proof is completed by applying the square root trick once again for $P_p[T^-] = P_p[T^+]$. \qed

Continuing the study of bond percolation at the critical probability, we next establish lower- and upper-bounds on the probability that the open cluster containing the origin extends past $B(n)$.

**Theorem 5 (Power law inequalities.)** There exist constants $A$ and $\alpha$ such that

$$\frac{1}{2} n^{-\frac{1}{2}} \leq P_p[0 \leftrightarrow \partial B(n)] \leq An^{-\alpha},$$

where $\partial B(n)$ denotes the border of $B(n)$.

**Proof:** For any $k \in [0, 2n - 1]$ let $A(k)$ denote the event that there is an open path from $(n, k)$ to the border of the $(2n + 1) \times (2n + 1)$ box centered at $(n, k)$ (which is nothing but a translation of $B(n)$).
Note that there are two disjoint paths of this kind when the box \([0,2n] \times [0,2n-1]\) has a left-to-right path of open bonds passing through \((n,k)\). Since a left-to-right open path in the box \([0,2n] \times [0,2n-1]\) has to cross the line \(x = n\) at least once, we get that

\[
P_{\frac{1}{2}}[LR(2n + 1, 2n)] \leq \sum_{k=0}^{2n-1} P_{\frac{1}{2}}[A(k) \circ A(k)].
\]

But \(P_{\frac{1}{2}}[LR(2n + 1, 2n)] = \frac{1}{2}\) by Lemma 1, and \(P_{\frac{1}{2}}[A(k) \circ A(k)] \leq P_{\frac{1}{2}}[A(k)]^2\) by the BK inequality. Since \(P_{\frac{1}{2}}[A(k)] = P_{\frac{1}{2}}[0 \leftrightarrow \partial B(n)]\), it follows that

\[
\frac{1}{2} \leq 2nP_{\frac{1}{2}}[0 \leftrightarrow \partial B(n)]^2,
\]

and this gives the left-hand side inequality.

To prove the right-hand side inequality we will rely again on duality. Note that if there is an open path from the origin to the boundary of \(B(n)\), then there is no closed circuit in each of the \((\frac{1}{2}, \frac{1}{2})\) centered annulæ of external radiæ \(3, 3^2, \ldots, 3^{\lfloor \log_3 n \rfloor - 1}\) of the dual lattice. But we know from the RSW theorem that each such annulus contains a closed circuit with some probability \(\xi > 0\), so

\[
P_{\frac{1}{2}}[0 \leftrightarrow \partial B(n)] \leq (1 - \xi)^{\lfloor \log_3 n \rfloor - 1} \leq (1 - \xi)^{\log_3 n - 2} = \frac{\tau^{\log_3 (1-\xi)}}{(1-\xi)^2}.
\]

\(\square\)