Compact Numerical Quadrature Formulas for Singular and Hypersingular Integrals and Integral Equations

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Abstract

We develop compact numerical quadrature formulas for finite-range integrals of the form $I[f] = \int_a^b f(x) \, dx$, where $f(x) = g(x)|x - t|^\beta$, $\beta$ being real, or $f(x) = g(x)(x - t)^\beta$, $\beta = -1, -3, \ldots$. These integrals are defined in the regular sense when convergent, and in the sense of Hadamard finite part when divergent. Assuming that $g \in C^\infty[a, b]$, or $g \in C^\infty(a, b)$ but can have arbitrary algebraic singularities at $x = a$ and/or $x = b$, and letting $h = (b - a)/n$, $n$ an integer, we derive asymptotic expansions (as $h \to 0$) for $T_n^* [f] = h \sum_{1 \leq j \leq n-1, x_j \neq t} f(x_j)$, where $x_j = a + jh$ and $t \in \{x_1, \ldots, x_{n-1}\}$. These asymptotic expansions are based on some recent generalizations of the Euler–Maclaurin expansion due to the author [A. Sidi, “Euler–Maclaurin expansions for integrals with arbitrary algebraic endpoint singularities.” Math. Comp., 81 (2012) pp. 2159–2173], and are used to construct our quadrature formulas, whose accuracies are then increased at will by applying to them the Richardson extrapolation process. We pay particular attention to the cases in which $\beta = -1, -2$ and $f(x)$ is periodic with period $T = b - a$ and $f \in C^\infty(-\infty, \infty) \setminus \{t + kT\}_{k=-\infty}^\infty$, which arises in the context of periodic singular and hypersingular integral equations. For this case, we propose the remarkably simple and compact quadrature formulas $\tilde{Q}_n[f] = h \sum_{j=1}^n f(t + jh - h/2)$ (for $\beta = -1$) and $\hat{Q}_n[f] = h \sum_{j=1}^n f(t + jh - h/2) - \pi^2 g(t)h^{-1}$ (for $\beta = -2$). We show that these formulas have spectral accuracy. We also show that each one of these formulas is exact for a family of singular integrals involving trigonometric polynomials. We also show how they can be used for solving singular and hypersingular integral equations in an efficient manner. We provide numerical examples that confirm the theory.