Lecture 14. Section 8.5. Integration of Rational functions by partial fractions.

Idea

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1} + \frac{3}{x-3}\,dx$$

$$= 2\cdot \ln|x+1| + 3\cdot \ln|x-3| + C.$$

Step 1.

If the leading coefficient of the denominator is not "1", factor it out.

Step 2.

If the degree of the numerator is greater or equal to the degree of the denominator, carry out long division.

Ex $\displaystyle\int \frac{2x^3-4x^2-x-3}{2x^2-4x-6}\,dx.$

$$\frac{2x^3-4x^2-x-3}{2x^2-4x-6} = \frac{1}{2}\cdot\frac{2x^3-4x^2-x-3}{x^2-2x-3}$$

$$\left(\begin{array}{r|l} & \;\; 2x \\ \hline x^2-2x-3 & \;\; 2x^3-4x^2-x-3 \\ & \;\; 2x^3-4x^2-6x \\ \hline & \qquad\qquad 5x-3. \end{array}\right)$$

$$= \frac{1}{2}\cdot\left(2x + \frac{5x-3}{x^2-2x-3}\right).$$

$$= \frac{1}{2}\left(2x + \frac{5x-3}{(x-3)\cdot(x+1)}\right).$$

Step 3. Factor the denominator completely into linear and irreducible quadratic terms.

Ex

$$\int \frac{4x^4-8x^3+8x^2-10x+8}{3x^4-6x^3+6x^2-6x+3}\,dx$$

$$\frac{4x^4-8x^3+8x^2-10x+8}{3x^4-6x^3+6x^2-6x+3} = \frac{1}{3}\cdot\frac{4x^4-8x^3+8x^2-10x+8}{x^4-2x^3+2x^2-2x+1}$$

$$\left(\begin{array}{r|l} & 4 \\ \hline x^4-2x^3+2x^2-2x+1 & 4x^4-8x^3+8x^2-10x+8 \\ & 4x^4-8x^3+8x^2-8x+4 \\ \hline & -2x+4 \end{array}\right)$$

$$= \frac{1}{3}\left(4+\frac{-2x+4}{x^4-2x^3+2x^2-2x+1}\right)$$

$$x^4-2x^3+2x^2-2x+1. \qquad \frac{p}{q}$$

$$x=1 \Rightarrow \quad 1-2+2-2+1=0$$

$$\begin{array}{r|l} & x^3-x^2+x-1 \\ \hline x-1 & x^4-2x^3+2x^2-2x+1. \\ & x^4-x^3 \\ \hline & -x^3+2x^2-2x+1 \\ & -x^3+x^2 \\ \hline & x^2-2x+1. \\ & x^2-x \\ \hline & -x+1 \\ & -x+1 \\ \hline & 0. \end{array}$$

$$x^4-2x^3+2x^2-2x+1$$

$$= (x-1)(x^3-x^2+x-1).$$

$$= (x-1)^2(x^2+1).$$

$$= \frac{1}{3}\left(4 + \frac{-2x+4}{(x-1)^2(x^2+1)}\right).$$

Ex Perform step 1-3 on

$$\frac{x^3+4x^2}{2x^2+8x-10} = \frac{1}{2}\cdot\frac{x^3+4x^2}{x^2+4x-5}.$$

$$\left(\begin{array}{r|l} & \quad x \\ \cline{2-2} x^2+4x-5 & x^3+4x^2 \\ & x^3+4x^2-5x \\ \cline{2-2} & \qquad\qquad 5x \end{array}\right)$$

$$= \frac{1}{2}\left(x + \frac{5x}{x^2+4x-5}\right)$$

$$= \frac{1}{2}\left(x + \frac{5x}{(x+5)(x-1)}\right).$$

Step 4.

① Linear form $(x-a)^k$

$$\rightsquigarrow \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}.$$

e.g. $$\frac{4}{\underbrace{(x-1)^2}_{a=1\ \ k=2}\cdot(x^2+1)^2} \rightsquigarrow \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}.$$

② Irreducible quadratic term $(x^2+bx+c)^m$

$$\rightsquigarrow \frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_mx+C_m}{(x^2+bx+c)^m}.$$

e.g. $$\frac{4}{(x-1)^2\cdot\underbrace{(x^2+1)^2}_{b=0,\ c=1,\ m=2}} \rightsquigarrow \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}.$$

Step 5.

Set the original function equal to sum of all the above

Solve for all the constants.

- Multiply everything by the common denominator

- Equate the coefficients of corresponding powers of x and solve the resulting equations.

Ex

$$\frac{4}{(x-1)^2(x^2+1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1x_1+C_1}{x^2+1} + \frac{B_2x_2+C_2}{(x^2+1)^2}.$$

mutiply. $(x-1)^2(x^2+1)^2$

$$\Rightarrow 4 = A_1 \cdot (x-1)(x^2+1)^2$$

$$+ A_2 (x^2+1)^2$$

$$+ (B_1x_1+C_1)(x-1)^2(x^2+1)$$

$$+ (B_2x_2+C_2)(x-1)^2.$$

$$\Rightarrow 4 = A_1 \left(x^5 - \boxed{x^4} + 2x^3 - 2x^2 + x - 1 \right)$$

$$+ A_2 \left(\boxed{x^4} + 2x^2 + 1 \right)$$

$$+ B_1 x^5 + (-2B_1 + C_1) \boxed{x^4} + (2B_1 - 2C_1) x^3 + (-2B_1 + 2C_1) x^2 + (B_1 - 2C_1) x + C_1 .$$

$$+ B_2 x^3 + (-2B_2 + C_2) x^2 + (B_2 - 2C_2) x + C_2$$

$$0 \cdot x^5 + 0 \cdot x^4 + 0 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 4$$

x^5: $A_1 + B_1 = 0.$

x^4: $-A_1 + A_2 - 2B_1 + C_1 = 0$

x^3: $2A_1 + 2B_1 - 2C_1 + B_2 = 0$

x^2: $-2A_1 + 2A_2 - 2B_1 + 2C_1 - 2B_2 + C_2 = 0.$

x: $A_1 + B_1 - 2C_1 + B_2 - 2C_2 = 0.$

1: $-A_1 + A_2 + C_1 + C_2 = 4$

$A_1 = -2,$

$A_2 = 1$

$B_1 = 2 \qquad C_1 = 1$

$B_2 = 2 \qquad C_2 = 0.$

$$\frac{4}{(x-1)^2 (x^2+1)^2} = \frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1} + \frac{2x}{(x^2+1)^2} .$$