Ex $\int \frac{30x}{(x+5)(x-1)}\,dx$.

$(x+5)' \rightsquigarrow \frac{A}{x+5}$.

$(x-1) \rightsquigarrow \frac{B}{x-1}$

$$\frac{30x}{(x+5)(x-1)} = \frac{A}{x+5} + \frac{B}{x-1}$$

multiply $(x+5)(x-1)$

$\Rightarrow 30x = A(x-1) + B(x+5)$

x: $30 = A+B$.

1: $0 = -A+5B$.

$\Rightarrow 30 = 6B$

$\Rightarrow B = 5. \quad A = 25.$

$$\Rightarrow \frac{30x}{(x+5)(x-1)} = \frac{25}{x+5} + \frac{5}{x-1}.$$

$$\Rightarrow \int \frac{30x}{(x+5)(x-1)}\,dx$$

$$= 25\int \frac{1}{x+5}\,dx + 5\int \frac{1}{x-1}\,dx$$

$$= 25 \cdot \ln|x+5| + 5 \cdot \ln|x-1| + C.$$

Ex $\int \frac{x^2-1}{x(x^2+1)^2}\,dx$

$x \rightsquigarrow \frac{A}{x}$

$(x^2+1)^2 \rightsquigarrow \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$$\frac{x^2-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

multiply $x(x^2+1)^2$

$$\Rightarrow x^2-1 = A(x^4+2x^2+1)$$

$$+ (Bx+C)(x^3+x)$$

$$+ (Dx+E)(x)$$

$$= A(x^4+2x^2+1)$$

$$+ Bx^4 + Cx^3 + Bx^2 + Cx$$

$$+ Dx^2 + Ex$$

x^4 : $0 = A+B$

x^3 : $0 = C$

x^2 : $1 = 2A+B+D$

x : $0 = C+E$

1 : $-1 = A$

$A=-1$, $C=0$, $E=0$. $B=1$, $D=2$.

$$\int \frac{x^2-1}{x(x^2+1)^2}\,dx$$

$$= -\int \frac{1}{x}\,dx + \boxed{\int \frac{x}{x^2+1}\,dx} + \boxed{\int \frac{2x}{(x^2+1)^2}\,dx}$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\cdot \ln|u| + C$$

$$\left[\begin{array}{l} x^2+1 = u \\ 2x\,dx = du \end{array}\right] \qquad = \frac{1}{2}\ln(x^2+1) + C.$$

$$\int \frac{2x}{(x^2+1)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$$

$$\left[\begin{array}{l} x^2+1 = u \\ 2x\,dx = du \end{array}\right] \qquad = -\frac{1}{1+x^2} + C.$$

$$\int \frac{x^2-1}{x(x^2+1)^2}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln(x^2+1) - \frac{1}{1+x^2} + C$$

Ex $\int \frac{2x-1}{x^2(x-2)^2}\,dx.$

$$x \rightsquigarrow \frac{A}{x} + \frac{B}{x^2}$$

$$x-2 \rightsquigarrow \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$\frac{2x-1}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

multiply $x^2(x-2)^2$

$$\Rightarrow \quad 2x-1 = A(x^3-4x^2+4x)$$
$$+ B(x^2-4x+4)$$
$$+ C(x^3-2x^2) + Dx^2$$

x^3 : $0 = A + C$

x^2 : $0 = -4A + B + D - 2C$

x : $2 = 4A - 4B$

1 : $-1 = 4B$.

$B = -\frac{1}{4}$, $A = \frac{1}{4}$, $C = -\frac{1}{4}$, $D = \frac{3}{4}$

$$\int \frac{2x-1}{x^2(x-2)^2}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx - \frac{1}{4}\int \frac{1}{x^2}\,dx$$

$$- \frac{1}{4}\int \frac{1}{x-2}\,dx + \frac{3}{4}\int \frac{1}{(x-2)^2}\,dx.$$

$$= \frac{1}{4}\ln|x| + \frac{1}{4}\frac{1}{x} - \frac{1}{4}\ln|x-2|$$

$$- \frac{3}{4}\cdot\frac{1}{(x-2)} + C.$$

Ex $\int \frac{\sin\theta}{\cos^2\theta + \cos\theta - 2}\,d\theta$.

$[\cos\theta = u \quad -\sin\theta\,d\theta = du]$

$$= \int \frac{-1}{u^2+u-2}\,du$$

$$= -\int \frac{1}{(u+2)(u-1)}\,du$$

$$\frac{1}{(u+2)(u-1)} = \frac{A}{u+2} + \frac{B}{u-1}$$

$$\Rightarrow \quad 1 = A(u-1) + B(u+2)$$

$\rightarrow A + B = 0 \quad \Rightarrow \quad B = 1/3, \; A = -1/3$

$-A + 2B = 1$

$$-\int \frac{1}{(u+2)(u-1)}\,du$$

$$= +\frac{1}{3}\int \frac{1}{u+2}\,du - \frac{1}{3}\int \frac{1}{u-1}\,du$$

$$= \frac{1}{3}\ln|u+2| - \frac{1}{3}\ln|u-1| + C$$

$$= \frac{1}{3}\ln|\cos\theta+2| - \frac{1}{3}\ln|\cos\theta-1| + C$$

Ex $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$A=-2,\ B=1,\ C=2,\ D=1.$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = -2\cdot\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx + \boxed{\int \frac{2x+1}{x^2+1}\,dx.}$$

$$\int \frac{2x+1}{x^2+1}\,dx = \underbrace{\int \frac{2x}{x^2+1}\,dx}_{x^2+1=u} + \underbrace{\int \frac{1}{x^2+1}\,dx}_{=\tan^{-1}x}.$$

Ex.

$$\frac{X^2+1}{(x-1)\cdot(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

other metod.

① multiply $(x-1)$

$$\frac{X^2+1}{(x-2)(x-3)} = A + \frac{B}{(x-2)}\cdot(x-1) + \frac{C}{(x-3)}\cdot(x-1)$$

put $x=1$

$$\frac{2}{-1\cdot(-2)} = A \Rightarrow A=1.$$

② multiply $(x-2)$

$$\frac{X^2+1}{(x-1)(x-3)} = \frac{A}{(x-1)}\cdot(x-2) + B + \frac{C}{(x-3)}\cdot(x-2)$$

put $x=2$

$$\frac{5}{1\cdot(-1)} = B \Rightarrow B=-5.$$

③ multiply $(x-3)$

$$\frac{X^2+1}{(x-1)\cdot(x-2)} = \frac{A}{(x-1)}\cdot(x-3) + \frac{B}{(x-2)}\cdot(x-3) + C.$$

put $x=3$

$$\frac{10}{2\cdot 1} = C \Rightarrow C=5.$$

Ex $\quad \frac{x-1}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$

$$\Rightarrow (x-1) = A(x+1)^2 + B(x+1) + C.$$

put $x=-1$

$$-2 = C$$

diff

$$\Rightarrow 1 = 2A(x+1) + B$$

put $x=-1$

$$1 = B$$

Diff

$$\Rightarrow 0 = 2A \Rightarrow A=0.$$