KEY (G)

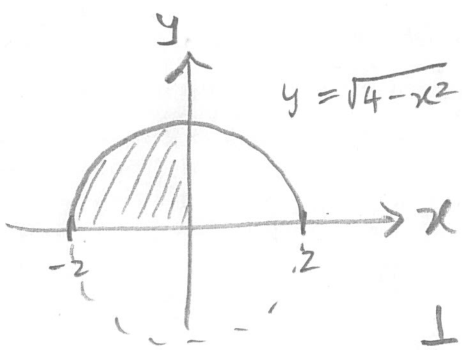
1. Short answer section: fill in the equality with the correct expression. (5 pts. each)

(a) If $F'(x) = f(x)$, then $\int f(g(x))g'(x)\,dx = \boxed{F(g(x)) + C}$

(b) From the FTC we have $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \boxed{f(g(x))\cdot g'(x)}$

2. Evaluate the integral using any method, but clearly show your work for full credit. *Hint: draw a picture.* (10 pts.)

$$\int_{-2}^{0} \sqrt{4-x^2}\,dx$$



area is $\frac{1}{4}$ area of circle of radius $r=2$. $A = \pi r^2$

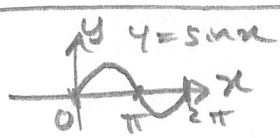
$\frac{1}{4}\pi(2)^2 = \boxed{\pi}$

3. Evaluate the integral. (14 pts.)

$$\int_0^{\pi/3} \cos(3x)\sin(3x)\,dx$$

u-sub

$$\boxed{\begin{aligned} u &= \sin(3x) \\ du &= 3\cos(3x)\,dx \end{aligned}}$$



$\sin(\pi) = 0$ ✓

$\sin(0) = 0$ ✓

$$= \int_*^* \tfrac{1}{3} u \, du$$

$$= \frac{1}{6} u^2 \Big|_*^*$$

$$= \frac{1}{6}\sin^2(3x) \Big|_0^{\pi/3}$$

$$= \frac{1}{6}\sin^2\left(3\cdot\frac{\pi}{3}\right) - \frac{1}{6}\sin^2(3\cdot 0)$$

$$= 0 - 0 = \boxed{0}$$

4. Solve the separable differential equation. (14 pts.)

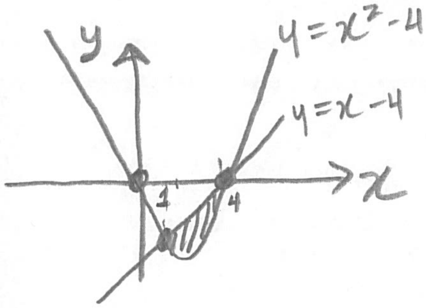
$$\frac{dy}{dx} = \frac{(y^2+1)e^{\sqrt{x}}}{2\sqrt{x}}$$

$$\int \frac{1}{y^2+1}\,dy = \int \frac{1}{2\sqrt{x}} e^{\sqrt{x}}\,dx = \int e^u\,du = e^{\sqrt{x}} + C$$

$$\tan^{-1}(y) = e^{\sqrt{x}} + C \implies \boxed{y = \tan\left(e^{\sqrt{x}} + C\right)}$$

5. Find the area bounded by the curves $y = x^2 - 4x$ and $y = x - 4$. Show all your work for credit.

(14 pts.)



$y = x(x-4)$

int. pts.

$x^2 - 4x = x - 4$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0$

$x = \underline{4, 1}$

$1 \quad 2 \quad 4$

quad. $(2)^2 - 4(2) = -4$ BOT

lin. $2 - 4 = -2$ TOP

$$\text{Area} = \int_1^4 \left[(x-4) - (x^2 - 4x)\right] dx = \int_1^4 (-x^2 + 5x - 4)\, dx = \frac{27}{6} = \boxed{\frac{9}{2}}$$

$$= -\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x \Big|_1^4 = \left(-\frac{64}{3} + \frac{80}{2} - 16\right) - \left(-\frac{1}{3} + \frac{5}{2} - 4\right) = \frac{-63 \cdot 2 + 75 \cdot 3 - 12 \cdot 6}{6}$$

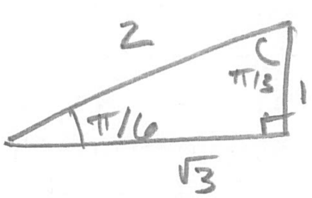
6. Let $y = \int_0^{\tan x} \cos(\pi t^2)\, dt$. Find the value of $\frac{dy}{dx}$ when $x = \pi/6$.

(14 pts.)

$$\frac{dy}{dx} = \cos(\pi \tan^2 x) \cdot \sec^2 x$$

$$\frac{dy}{dx}\Big|_{x=\pi/6} = \cos(\pi \tan^2(\pi/6)) * \sec^2(\pi/6)$$

$$= \cos\left(\pi (1/\sqrt{3})^2\right) \cdot \left(\frac{2}{\sqrt{3}}\right)^2$$



$$= \cos(\pi/3) \cdot \frac{4}{3}$$

$$= \frac{1}{2} \cdot \frac{4}{3} = \boxed{2/3}$$

7. Evaluate the limit using any method. Clearly indicate your work for full credit. (10 pts.)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(3\left(1+\frac{2i}{n}\right)^2+1\right)$$

$$= \int_1^3 (3x^2+1)\,dx = x^3+x \Big|_1^3$$

$$= (27+3) - (1+1)$$

$$= \boxed{28}$$

8. Find a closed formula for L_n the left-endpoint Riemann sum for the function $y = x^2 - 2$ over the interval $[0, 3]$. (14 pts.)

$\Delta x = 3/n$

$x_i = 3i/n$

$f(x) = x^2 - 2$

$L_n = \sum_{i=0}^{n-1} \Delta x\, f(x_i)$

$$L_n = \sum_{i=0}^{n-1} \Delta x \cdot f(x_i) = \sum_{i=0}^{n-1} \frac{3}{n} \cdot \left(\left(\frac{3i}{n}\right)^2 - 2\right)$$

$$= \frac{27}{n^3} \sum_{i=0}^{n-1} i^2 - \frac{6}{n} \cdot \sum_{i=0}^{n-1} 1$$

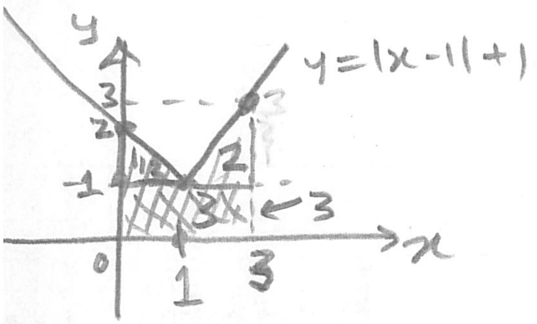
$$= \frac{27}{n^3} \frac{n(n-1)(2n-1)}{6} - 6$$

$$\Rightarrow \lim_{n\to\infty} L_n = 27 \cdot \frac{2}{6} - 6 = 9 - 6 = \boxed{3} \checkmark$$

Check $\int_0^3 x^2 - 2\,dx = \frac{1}{3}x^3 - 2x \Big|_0^3 = \frac{1}{3}3^3 - 6 - 0 = 9 - 6 = \boxed{3} \checkmark$

Bonus: *(up to +5pts.)* Find the average value of $y = |x-1|+1$ over the interval $[0,3]$.

$$\frac{1}{3-0}\int_0^3 (|x-1|+1)\,dx = \frac{1}{3}\int_0^3 (x-1)+1\,dx$$



$$= \frac{1}{3}\int_0^1 -(x-1)+1\,dx + \frac{1}{3}\int_1^3 (x-1)+1\,dx$$

$$= \frac{1}{3}\int_0^1 -x+2\,dx + \frac{1}{3}\int_1^3 x\,dx$$

$$= \frac{1}{3}\left(-\frac{1}{2}x^2+2x\right)\Big|_0^1 + \frac{1}{6}x^2\Big|_1^3$$

area

$5 + 1/2 = 11/2$

$$= \frac{1}{3}\left(-\frac{1}{2}+2-0\right)+\frac{1}{6}(9-1)$$

avg value

$\frac{\text{area}}{\text{interval}} = \frac{11/2}{3}$

$$= \frac{1}{3}\cdot\frac{3}{2}+\frac{8}{6}$$

$$= \frac{1}{2}+\frac{4}{3} = \frac{3+8}{6} = \boxed{\frac{11}{6}}$$

$= \boxed{11/6}$