EXAM 2 - E

1. Evaluate the integrals. (15pts. each)

(a) $\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - \int 2\sqrt{x}\cdot\frac{1}{x}\,dx$

IBP - box

$u = \ln x \qquad dv = \frac{1}{\sqrt{x}}\,dx$

$du = \frac{1}{x}\,dx \qquad v = 2\sqrt{x}$

$= 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx$

$= \boxed{2\sqrt{x}\ln x - 4\sqrt{x} + C}$

(b) $\int \frac{1}{\sqrt{9x^2-4}}\,dx = \int \frac{1}{\sqrt{4\sec^2\theta - 4}}\cdot\frac{2}{3}\sec\theta\tan\theta\,d\theta$

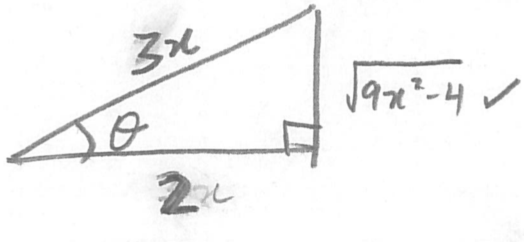
trig-sub BOX

$x = \frac{2}{3}\sec\theta$

$dx = \frac{2}{3}\sec\theta\tan\theta\,d\theta$

$= \frac{\cancel{2}}{3}\int \frac{\sec\theta\,\cancel{\tan\theta}}{\cancel{2\tan\theta}}\,d\theta$

$\sec\theta = \frac{3x}{2}$



$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$

$= \boxed{\frac{1}{3}\ln\left|\frac{3x}{2} + \frac{\sqrt{9x^2-4}}{2}\right| + C}$

2. Evaluate the integral. (15 pts.)

$$\int \sin^2(t)\cos^2(t)\,dt$$

$$= \int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{1}{4}\int (1-\cos^2 2t)\,dt = \frac{1}{4}\int \left(1-\left(\frac{1+\cos 4t}{2}\right)\right)dt$$

$$= \frac{t}{8} - \frac{1}{8}\cdot\frac{1}{4}\sin(4t) + C = \boxed{\frac{t}{8} - \frac{1}{32}\sin(4t) + C}$$

3. Use the formula below to find the theoretical error estimate for approximating the area under $f(x) = \frac{1}{x^2}$ using the trapezoid rule over the interval $[2, 10]$ using $n = 8$ trapezoids. (10 pts.)

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}, \quad |f''(x)| \leq M \text{ over } [a,b]$$

$M \geq |f''(x)|$ over $[2,10]$

$f'(x) = \frac{-2}{x^3}$

$f''(x) = \frac{6}{x^4} \leq \frac{6}{2^4} = \frac{3}{8} = M$

$$|E_T| \leq \frac{(3/\cancel{8})(10-2)^3}{12 \cdot \cancel{8^2}}$$

$$= 3/12 = \boxed{1/4}$$

$$|E_T| \leq 1/4$$

4. Determine if the given improper integral converges or diverges. Do not evaluate. Fully justify your answer for full credit. (15 pts.)

$$\int_3^\infty \frac{\sqrt{x^2+1}}{x^3+x}\,dx = \int_3^\infty \frac{\sqrt{x^2+1}}{x(x^2+1)}\,dx$$

$$= \int_3^\infty \frac{1}{x\sqrt{x^2+1}}\,dx$$

$$\leq \int_3^\infty \frac{1}{x\sqrt{x^2}}\,dx$$

$$= \int_3^\infty \frac{1}{x^2}\,dx$$

$< \infty$ p-test w/ $p=2>$

Converges by DCT

Since $f(x) \leq \frac{1}{x^2} = g(x)$

and $\int_3^\infty g(x)\,dx$ coverges

by p-test w/ $p=2$

5. Let $a_n = \left(1+\frac{3}{n}\right)^{2n}$. Determine the limit $\lim_{n\to\infty} a_n$ or show that it diverges. If you use a formula, state the formula you are using, and otherwise fully justify your answer for full credit. (15 pts.)

$$y_n = \ln\left(\left(1+\frac{3}{n}\right)^{2n}\right)$$

$$= 2n \ln(1+3/n)$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} 2n \cdot \ln(1+3/n)$$

$$= \lim_{n\to\infty} \frac{\ln(1+3/n)}{1/2n}$$

L'Hop, 0/0

$$= \lim_{n\to\infty} \frac{\frac{1}{1+3/n} * -3/\cancel{n^2}}{-1/2\cancel{n^2}} = \lim_{n\to\infty} 6\cdot\frac{1}{1+3/n} = \underline{6}$$

Since $y_n \to 6$

$$e^{y_n} = e^{\ln((1+3/n)^{2/n})} \to e^6$$

$$\Rightarrow (1+3/n)^{2/n} \to \boxed{e^6}$$

6. Evaluate the integral. (15 pts.)

$$\int \frac{-2x+1}{x^2(x^2+1)}\,dx$$

$$\frac{-2x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow -2x+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$\Rightarrow -2x+1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$= x^3(A+C) + x^2(B+D) + Ax + B$$

$$\Rightarrow \boxed{\begin{array}{l} A=-2 \\ B=1 \\ C=2 \\ D=-1 \end{array}}$$

So $\text{Integral} = \int \left(\frac{-2}{x} + \frac{1}{x^2} + \frac{2x-1}{x^2+1} \right) dx$

$$= \boxed{-2\ln x - \frac{1}{x} + \ln(x^2+1) - \tan^{-1} x + C}$$

Bonus: *(+1pts. each)* Give an example of a sequence which satisfies the condition or say "impossible". No work is required.

(many answers possible)

(a) a_n which is increasing, bounded, and converges. $a_n = 1 - 1/n$

(b) b_n which is decreasing and unbounded. $b_n = -n$

(c) c_n which is non-monotone and bounded, which converges. $c_n = \frac{(-1)^n}{n}$

(d) d_n which converges to $\frac{1}{e}$. $d_n = \frac{1}{e} + \frac{1}{n}$

(e) e_n which is increasing, unbounded, and converges.

impossible