Exam 2 - G

1. Evaluate the integrals. (15pts. each)

(a) $\displaystyle\int xe^{3x}\,dx = \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx = \boxed{\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C}$

IBP box

$$\boxed{\begin{array}{ll} u = x & dv = e^{3x}\,dx \\ du = dx & v = \frac{1}{3}e^{3x} \end{array}}$$

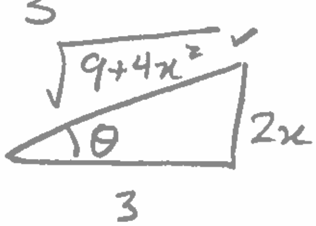
(b) $\displaystyle\int \frac{1}{\sqrt{9+4x^2}}\,dx = \int \frac{1}{\sqrt{9+4\left(\frac{3}{2}\tan\theta\right)^2}}\,\frac{3}{2}\sec^2\theta\,d\theta$

trig-sub Box

$$\boxed{\begin{array}{l} x = \frac{3}{2}\tan\theta \\ dx = \frac{3}{2}\sec^2\theta\,d\theta \end{array}}$$

$$= \int \frac{\frac{3}{2}\sec^2\theta}{\sqrt{9+4\cdot\frac{9}{4}\tan^2\theta}}\,d\theta$$

$\frac{2x}{3} = \tan\theta$



$$= \int \frac{\frac{3}{2}\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta = \int \frac{\frac{3}{2}\sec^2\theta}{\sqrt{9\sec^2\theta}}\,d\theta$$

$$= \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\frac{1}{2}\ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{2x}{3}\right| + C}$$

2. Evaluate the integral. (15 pts.)

$$\int \sin^2(t)\cos^2(t)\,dt$$

$$= \int \left(\frac{1-\cos(2t)}{2}\right)\left(\frac{1+\cos(2t)}{2}\right) dt$$

$$= \frac{1}{4}\int (1-\cos^2(2t)\ dt)$$

$$= \frac{1}{4}\int \left[1-\left(\frac{1+\cos(4t)}{2}\right)\right] dt$$

$$= \frac{1}{4}\int \left[\frac{1}{2} - \frac{1}{2}\cos(4t)\right] dt$$

$$= \int \left(\frac{1}{8} - \frac{1}{8}\cos(4t)\right) dt$$

$$= \boxed{\frac{t}{8} - \frac{1}{32}\sin(4t) + C}$$

3. Use the formula below to find the theoretical error estimate for approximating the area under $f(x) = \dfrac{1}{x}$ using the trapezoid rule over the interval $[4, 8]$ using $n = 4$ trapezoids. (10 pts.)

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}, \quad |f''(x)| \leq M \text{ over } [a, b].$$

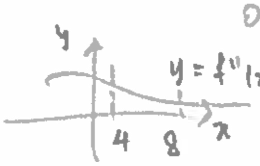
$f(x) = 1/x$

$f'(x) = -1/x^2$

$f''(x) = 2/x^3$

$f'''(x) = \dfrac{-6}{x^4} < 0$

So $f''(x) > 0$ and decreasing over $[4, 8]$



Hence $M = f''(4) = \dfrac{2}{4^3} = \dfrac{1}{32}$

So

$$|E_T| \leq \frac{(1/32)\cdot(8-4)^{\cancel{3}}}{12\cancel{(4)^2}} = \frac{1}{8}\cdot\frac{1}{12} = \frac{1}{96}$$

$$\boxed{|E_T| \leq \frac{1}{96}}$$

4. Determine if the given improper integral converges or diverges. Do not evaluate. Fully justify your answer for full credit. (15 pts.)

$$\int_2^{\infty} \frac{\sqrt{x}}{(x+1)^{5/3}}\, dx$$

$\frac{5}{3} - \frac{1}{2} = \frac{10-3}{6} = 7/6 > 1$

$f(x) = \frac{\sqrt{x}}{(x+1)^{5/3}} \leq \frac{\sqrt{x}}{x^{5/3}} = \frac{1}{x^{5/3 - 1/2}} = \frac{1}{x^{7/6}} = g(x).$

Since $\int_2^{\infty} \frac{1}{x^{7/6}}\, dx$ converges by p-test w/ $p = 7/6 > 1$,

and $\frac{1}{x^{7/6}} \geq f(x)$ over $[2, \infty)$,

the integral $\boxed{\int_2^{\infty} f(x)\, dx \text{ also converges by DCT.}}$

5. Let $a_n = \left(1 + \frac{2}{n}\right)^{4n}$. Determine the limit $\lim_{n\to\infty} a_n$ or show that it diverges. If you use a formula, state the formula you are using, and otherwise fully justify your answer for full credit. (15 pts.)

Let $a_n = \left(1 + \frac{2}{n}\right)^{4n}$

Then $\ln(a_n) = 4n \cdot \ln\left(1 + \frac{2}{n}\right)$

$= \frac{\ln(1 + 2/n)}{1/4n}$

So $\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} \frac{\ln(1+\frac{2}{n})}{1/4n}$

$\overset{\text{L'Hop}}{\underset{0/0}{=}} \lim_{n\to\infty} \frac{\frac{1}{1+2/n} \cdot -2/n^2}{-1/4n^2} = \lim_{n\to\infty} \frac{8}{1+2/n} = \underline{\underline{8}}$

So $\ln(a_n) \to 8$

hence

$a_n = e^{\ln(a_n)} \longrightarrow e^8$

So $\boxed{a_n \to e^8}$

6. Evaluate the integral. (15 pts.)

$$\int \frac{x-1}{x^2(x^2+1)}\,dx$$

$$\frac{x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow x-1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$\Rightarrow x-1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$\Rightarrow x-1 = x^3(A+C) + x^2(B+D) + x(A) + B$$

$$\Rightarrow \boxed{\begin{array}{l} A=1 \\ B=-1 \\ C=-1 \\ D=1 \end{array}}$$

So

$$\int \frac{x-1}{x^2(x^2+1)}\,dx = \int \frac{1}{x} - \frac{1}{x^2} + \frac{-x+1}{x^2+1}\,dx = \int \frac{1}{x} - \frac{1}{x^2} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,d$$

$$= \boxed{\ln x + \frac{1}{x} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C}$$

Bonus: *(+1pts. each)* Give an example of a sequence which satisfies the condition or say "impossible". No work is required. many answers possible, except for (e)

(a) a_n which is decreasing, unbounded, and diverges.

$a_n = -n$

(b) b_n which is non-monotone, bounded, and converges.

$b_n = \frac{(-1)^n}{n}$

(c) c_n which is increasing, bounded, and converges.

$c_n = 1 - 1/n$

(d) d_n which converges to e^3.

$d_n = e^3 - 1/n$

(e) e_n which is increasing, unbounded, and converges.

impossible.