Exam 3 - E

1. Determine if the given series converge or diverge. Fully justify your answer by providing *(a)* coverges/diverges, *(b)* the name of the test you used, and *(c)* supporting work for the answer based on the test you have chosen. (15pts. each)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(2n+1)!}$

ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{\cancel{n}+1}}{(2n+3)!} * \frac{(2n+1)!}{\cancel{3^n}}$$

$$= \lim_{n\to\infty} \frac{3}{(2n+3)(2n+2)} = 0 = \rho$$

Since $\rho < 1$ by the ratio test

The series $\sum a_n$ $\boxed{\text{converges}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2\sqrt{n}(sqrt{n} + 1)^{3/2}}$

integral test

$$\int_2^{\infty} \frac{1}{2\sqrt{x}(\sqrt{x}+1)^{3/2}}\,dx = \int_{\sqrt{2}+1}^{\infty} \frac{1}{u^{3/2}}\,du < \infty$$

by p-test w/ $p = 3/2 > 1$.

By the integral test the series $\sum a_n$

$\boxed{\text{Converges}}$

2. Find the **value** of the geometric series, or show that the series diverges. (15 pts.)

$$\sum_{n=2}^{\infty} \frac{2^n}{3^{2n+1}}$$

$$\sum_{n=2}^{\infty} \frac{2^n}{3^{2n}} \cdot \frac{1}{3} = \frac{1}{3} \sum_{n=2}^{\infty} \left(\frac{2}{9}\right)^n$$

$$= \frac{1}{3}\left(\frac{2}{9}\right)^2 \cdot \left(\frac{1}{1-2/9}\right) = \frac{1}{3} \cdot \frac{4}{81} \cdot \frac{1}{7/9} =$$

$$= \frac{1}{3} \cdot \frac{4}{\cancel{81}\,9} \cdot \frac{\cancel{9}}{7} = \frac{4}{21 \cdot 9}$$

$$= \boxed{\frac{4}{189}}$$

3. Determine if the given alternating series converges or diverges. (10 pts.)

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{n+3}{3n+1}\right)$$

The series $\sum (-1)^n a_n$ $\boxed{\text{diverges}}$ by the divergence test since

$$a_n = \frac{n+3}{3n+1} \not\to 0.$$

4. Determine if the alternating series converges absolutely, converges conditionally, or diverges. (15 pts.)

$$\sum_{n=2}^{\infty}(-1)^n \frac{n^2}{n^{7/3}-1}$$

absolute convergence?

$\sum a_n = \sum \frac{n^2}{n^{7/3}-1}$ LCT w/ $b_n = \frac{1}{n^{1/3}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^{7/3}-1} * n^{1/3} \overset{\text{L'Hop}}{=} 1 = L, \ (0 < L < \infty)$

And $\sum \frac{1}{n^{1/3}}$ diverges by p-test w/ $p = 1/3 \leq 1$, so $\sum a_n$ diverges

conditional convergence?

$\sum (-1)^n a_n$ converges by alt. series test since

a_n pos ✓ dec ✓ $a_n \to 0$ ✓

$\boxed{\text{conditionally converges}}$

5. Find the radius and interval of convergence of the given power series. (15 pts.)

$$\sum_{n=0}^{\infty} \frac{n^2}{n^2+1} x^n$$

ratio test

$\lim_{n\to\infty} \frac{a_{n+1}x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \frac{(n+1)^2 x^{n+1}}{n^2+2n+3} * \frac{n^2+1}{n^2 \cdot x^n} = \lim_{n\to\infty} \frac{(n^2+1)(n+1)^2}{(n^2+2n+3)n^2} \cdot x$

$\overset{\text{L'Hop}}{=} 1 \cdot x = \rho$

set $|\rho| < 1$

get $(-1, 1)$ unchecked endpoints.

$x = 1$ $\sum \frac{n^2}{n^2+1}$ diverges by divergence test

$x = -1$ $\sum \frac{n^2}{n^2+1}(-1)^n$ ditto

$\boxed{\text{interval } (-1,1) \text{ radius } R = 1}$

6. Find the Taylor series expansion at $x = 0$ for the function $f(x) = \frac{1}{9}e^{3x}$ (15 pts.)

B-rule $\frac{1}{9}e^{3x} = \frac{1}{9}\sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n = \boxed{\sum_{n=0}^{\infty} \frac{3^{n-2}}{n!}x^n}$

OR

$f(x) = \frac{1}{9}e^{3x}$

$f'(x) = \frac{1}{3}e^{3x}$

$f''(x) = e^{3x}$

$f^{(3)}(x) = 3e^{3x}$

$f^{(4)}(x) = 3^2 e^{3x}$

$\vdots$

$f^{(n)}(x) = 3^{n-2}e^{3x}$

So $\frac{f^{(n)}(0)}{n!} = \frac{3^{n-2}}{n!}$ and

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{3^{n-2}}{n!}x^n}$$

Bonus: *(+1pts. each)* Determine if the statement is True or False. No work is required.

(a) If $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges. TRUE/FALSE

(b) If $\lim_{n\to\infty} a_n = \ell$ and $\ell < 1$, then the series with terms a_n converges. TRUE/FALSE

(c) If $\sum a_n$ converges, then the terms a_n tend to zero. TRUE/FALSE

(d) If the alternating series $\sum(-1)^n a_n$ converges, and a_n satisfies all the requirements of the alternating series test, then the series $\sum a_n$ also converges. TRUE/FALSE

(e) If $\frac{a_{n+1}}{a_n} = 1$ for all n, then $\sum a_n$ diverges. TRUE/FALSE