Exam 3-G

1. Determine if the given series converge or diverge. Fully justify your answer by providing *(a)* coverges/diverges, *(b)* the name of the test you used, and *(c)* supporting work for the answer based on the test you have chosen. (15pts. each)

(a) $\sum_{n=1}^{\infty} \frac{5^n}{(2n)!}$

Ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{5^n} = \lim_{n\to\infty} \frac{5}{(2n+2)(2n+1)} = 0 = \rho$$

Since $\rho<1$ by the ratio test The series $\sum a_n$

$\boxed{\text{Converges}}$

(b) $\sum_{n=2}^{\infty} \frac{3}{n(\ln n)^{3/2}}$

Integral test

$$\int_2^\infty \frac{3}{x(\ln x)^{3/2}}\,dx = \int_{\ln 2}^{\infty} \frac{3}{u^{3/2}}\,du < \infty$$

by p-test w/ $p=3/2>1$.

So by the integral test the series $\sum a_n$ $\boxed{\text{Converges}}$

2. Find the **value** of the geometric series, or show that the series diverges. (15 pts.)

$$\sum_{n=2}^{\infty} \frac{3^{n-1}}{2^{2n+1}}$$

$$\sum_{n=2}^{\infty} \frac{3^n}{2^{2n}} \cdot \frac{1/3}{2} = \frac{1}{6} \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{6} \cdot \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{1-3/4}\right)$$

$$= \frac{3}{32}\left(\frac{1}{1/4}\right) = \boxed{\frac{3}{8}}$$

3. Determine if the given alternating series converges or diverges. (10 pts.)

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{2n+1}{2n+3}\right)$$

The series $\sum (-1)^n a_n$ $\boxed{\text{diverges}}$ by the divergence test since

$$a_n = \frac{2n+1}{2n+3} \not\to 0.$$

4. Determine if the alternating series converges absolutely, converges conditionally, or diverges. (15 pts.)

$$\sum_{n=2}^{\infty}(-1)^n\frac{n}{n^{3/2}-1}$$

absolute convergence? $\boxed{\text{NO}}$

$\sum a_n = \sum \frac{n}{n^{3/2}-1}$ LCT w/ $b_n = \frac{1}{n^{1/2}}$ L'Hop

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{3/2}-1} * n^{1/2} \stackrel{\text{L'Hop}}{=} 1 = L$ $(0<L<\infty$

and $\sum b_n$ diverges so $\sum a_n$ diverges as well.

conditional convergence? $\boxed{\text{yes}}$

$\sum(-1)^n a_n$ converges by alt. series test since

a_n pos ✓ dec ✓ $a_n \to 0$ ✓.

$\boxed{\text{Conditionally Converges}}$

5. Find the radius and interval of convergence of the given power series. (15 pts.)

$$\sum_{n=0}^{\infty}\frac{n}{n^2+1}x^n$$

ratio test

$\lim_{n\to\infty} \frac{a_{n+1}x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \frac{n+1}{n^2+2n+3} x^{n+1} * \frac{n^2+1}{nx^n}$

$= \lim_{n\to\infty} \frac{(n+1)(n^2+1)}{n(n^2+2n+3)} x \stackrel{\text{L'Hop}}{=} 1 \cdot x = \rho$

$\boxed{\text{interval } [-1,1) \quad \text{radius } R=1.}$

Set $|\rho|<1$ so $(-1,1)$ unchecked endpts.

$x=1$ $\sum \frac{n}{n^2+1}$ diverges by LCT w/ $b_n = 1/n$

$x=-1$ $\sum (-1)^n \frac{n}{n^2+1}$ converges by alt series test

6. Find the Taylor series expansion at $x = 0$ for the function $f(x) = \frac{1}{1+3x}$ (15 pts.)

$\mathcal{B}$-rule

$$\frac{1}{1-(-3x)} = \sum_{n=0}^{\infty} (-3x)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n 3^n \cdot x^n}$$

or

long way

$$f(x) = \frac{1}{1+3x}$$

$$f'(x) = \frac{-1}{(1+3x)^2} * 3$$

$$f''(x) = \frac{2}{(1+3x)^3} * 3^2$$

$\vdots$

$$f^{(n)}(x) = \frac{(-1)^n n!}{(1+3x)^n} * 3^n$$

So $\frac{f^{(n)}(0)}{n!} = (-1)^n * 3^n$

and

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^n \cdot 3^n x^n}$$

Bonus: *(+1pts. each)* Determine if the statement is True or False. No work is required.

(a) If $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges. FALSE

(b) If $\lim_{n\to\infty} a_n = \ell$ and $\ell < 1$, then the series with terms a_n converges. FALSE

(c) If $\sum a_n$ converges, then the terms a_n tend to zero. TRUE

(d) If the alternating series $\sum(-1)^n a_n$ converges, and a_n satisfies all the requirements of the alternating series test, then the series $\sum a_n$ also converges. FALSE

(e) If $\frac{a_{n+1}}{a_n} = 1$ for all n, then $\sum a_n$ diverges. TRUE.