MATH 1553, Intro to Linear Algebra
FINAL EXAM STUDY GUIDE

In studying for the final exam, you should FIRST study all tests and quizzes we have had this semester (solutions can be found on Canvas). Then go over class worksheets, old homework problems and examples/clicker questions we have done in class.

The final is comprehensive and will cover all the topics we have discussed this term. The general topics are listed below.

Gauss-Jordan Elimination (L 1.1-1.2)

Span and Linear Independence (L 1.3-1.5, 1.7)

Linear Transformations (L 1.8, 1.9)

Inverses, Elementary Matrices, and LU (L 2.1-2.3, 2.5)

Subspaces and Basis (L 2.8, 2.9)

Determinants (L 3.1-3.2)

Eigenvalues and Eigenvectors (L 5.1-5.3)

Complex Eigenvalues and Eigenvectors (L 5.5)

Orthogonalization and Gram-Schmidt (L 6.1-6.4)

Least Squares (L 6.5)

Practice Problems

1. Define the following terms: span, linear combination, linearly independent, linear transformation, column space, nullspace, transpose, inverse, elementary matrix, dimension, rank, nullity, determinant, eigenvalue, eigenvector, eigenspace, diagonalizable, orthogonal

2. Suppose that A is an $m \times n$ matrix.

(a) How do we determine the number of pivotal columns?

(b) What do the pivotal columns tell us about the solution to the equation $A\vec{x} = \vec{b}$?

(c) What space is equal to the span of the pivotal columns?

(d) Is it possible to write $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix? If not, what additional information would you need?

(e) Is it possible to write $A = QR$, where Q has orthonormal columns and R is an upper triangular matrix? If not, what additional information would you need?

(f) What is the difference between solving $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{0}$? How are these two solutions related geometrically?

(g) If $rank(A) = r$, where $0 < r \le n$, how many columns are pivotal? What is the dimension of the solution space to $A\vec{x} = \vec{0}$?

3. Suppose that T_A is a linear transformation $T_A : \Re^n \to \Re^m$ with associated matrix A.

(a) What are the dimensions of A?

(b) If $\vec{x} \in \Re^n$, how can we find $T_A(\vec{x})$?

(c) Using the matrix A, how would we know if T_A is one-to-one? Onto?

(d) How do we find the range of T_A?

4. Suppose A is an $n \times n$ invertible matrix.

(a) What can you say about the columns of A?

(b) What is $rank(A)$? $nullity(A)$?

(c) What do you know about $\det(A)$?

(d) How many solutions are there to the equation $A\vec{x} = \vec{b}$?

(e) What is the nullspace of A?

(f) Do you know anything about the eigenvalues of A?

(g) Do you know whether or not A is diagonalizable?

(h) How can we use the columns of A to create an orthogonal basis for $\Re^n$?

5. Suppose A is an $n \times n$ matrix with characteristic equation $p(\lambda) = \det(A - \lambda I)$.

(a) What is the degree of $p(\lambda)$?

(b) Counting multiplicities, how many eigenvalues will A have?

(c) If $p(0) = 0$, what do you know about the matrix A?

(d) How will you know if A is diagonalizable?

(e) If A is 3×3 and has a complex eigenvalue, how many real roots are there to $p(\lambda)$?

(f) Suppose $p(c) = 0$ for some real number c. How do you find the values of $\vec{x}$ for which $A\vec{x} = c\vec{x}$?

(g) If $\lambda_1, \lambda_2, ... \lambda_r$ are the distinct eigenvalues of A, counting multiplicities, what is the sum of the eigenvalues? The product?

(h) If A is NOT triangular or diagonal, do the solutions of $p(\lambda)$ change when A is reduced to echelon form? Why or why not?

6. Describe the span of the vectors $\begin{bmatrix} -6 \\ 7 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$. If the span is a line or a plane, give the equation of the line or plane.

7. Which of the following vectors form a linearly independent set?

$$\vec{v_1} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 6 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} -1 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

Write one of the vectors as a linear combination of the others.

8. Find the eigenvalues and bases for the eigenspaces for the following matrices.

(a) $A = \begin{bmatrix} 4 & -3 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

9. Find the least squares solution of the system

$$x + 2y = 0$$

$$2x + y + z = 1$$

$$2y + z = 3$$

$$x + y + z = 0$$

$$3x + 2z = -1.$$

10. Find the matrix for the linear transformation $T : \Re^3 \to \Re^3$ that first dilates by a factor of 5, then reflects about the yz-plane, then rotates counterclockwise about the y-axis by an angle of $60°$, and lastly projects onto the xz-plane.

11. Find A^{10} if $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

12. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be a basis for a vector space V. Use the Gram-Schmidt process to construct an orthogonal basis for V. If $A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}$, find matrices Q and R so that $A = QR$.

13. Find the determinant of the matrix

$$A = \begin{bmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}.$$

14. Given that $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}$, determine the following:

(a) a basis for the column space.

(b) a geometric description of the column space.

(c) a basis for the nullspace.

(d) a geometric description of the nullspace.

15. Find the numbers a, b, c, and d so that the matrix $\begin{bmatrix} 1 & 2 & 3 & | & a \\ 0 & 4 & 5 & | & b \\ 0 & 0 & d & | & c \end{bmatrix}$ has (a) no solution and (b) infinitely many solutions.

16. Celia has one hour to spend at the CRC, and she wants to jog, play handball, and ride a bicycle. Jogging uses 13 calories per minute, handball uses 11, and cycling uses 7 calories per minute. She jogs twice as long as she rides the bicycle. How long should she participate in each of these activities in order to use exactly 660 calories?

17. Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$. Then find matrices P and C so that $A = PCP^{-1}$, and write the matrix C as the product of a rotation and a scaling matrix.

Solutions

6. The span is the plane $24x + 30y - 33z = 0$.

7. $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent and $\vec{v_4} = 2\vec{v_1} - \vec{v_2} - \vec{v_3}$.

8. (a) Eigenvalues: $\lambda = 4, -2, 2$; Eigenvectors: $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

(b) Eigenvalues: $\lambda = -2, -2, 4$; Eigenvectors: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

9. $x = -\frac{187}{185}, y = \frac{137}{185}, z = \frac{43}{37}$

10. $[T] = \begin{bmatrix} -\frac{5}{2} & 0 & \frac{5\sqrt{3}}{2} \\ 0 & 0 & 0 \\ \frac{5\sqrt{3}}{2} & 0 & \frac{5}{2} \end{bmatrix}$

11. $A^{10} = \begin{bmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 1023 & 0 & 2047 \end{bmatrix}$

12. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \right\}$

$Q = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $R = \begin{bmatrix} 2 & 1.5 & 1 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

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14. (a) A basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

(b) The column space is a line with equation $x = t, y = 2t, z = -t$.

(c) A basis for the null space is $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) The null space is the plane with equation $x + 4y + 2z = 0$.

15. (a) $d = 0$ and $c \neq 0$; (b) $d = 0$ and $c = 0$. Note that a and b have no effect on these answers.

16. Solutions are $x = 2z, y = 60 - 3z$, and $0 \leq z \leq 20$.

17. $\lambda = 4 \pm i, C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}, P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and the scaling factor is $r = \sqrt{17}$.