RESEARCH STATEMENT

SALVADOR BARONE

1. Introduction

The origins of modern real algebraic geometry date back almost one hundred years to the solution of Hilbert's 17th Problem, but the recent development of the field in the last twenty years has been explosive. This development is closely related to interactions with several areas of pure mathematics including o-minimal geometry and model theory, computational and enumerative geometry, complexity theory, and the theory of moment problems and convex optimization. Recent work by Zahl [29] also establishes strong connections between results in real algebraic geometry and problems in harmonic analysis related to the Kakeya problem.

Real algebraic geometry is the study of semi-algebraic sets, subsets of R^k which are defined by a boolean combination of polynomial equalities and inequalities. Given a description of a semi-algebraic set $S \subset \mathrm{R}^k$, one type of quantitative question which arises is to bound the Betti numbers, the sum of the Betti numbers, or the Euler characteristic of S in terms of the maximum degree and number of the polynomials in the description of S. Theoretical bounds of this type are used to prove results in algorithmic and computational semi-algebraic geometry where, for example, one might wish to design an algorithm which can decide if two elements of a semi-algebraic set are in the same semi-algebraically connected component or not and, if so, to output a semi-algebraic path from one point to the other. Algorithms of this type have applications to robot motion planning and computer aided design. On the other hand, many of these quantitative aspects have been extended beyond the setting of semi-algebraic sets to the more general theory of o-minimal geometry, an axiomatic theory geared towards defining and exploring a tame geometry which shares many important properties with semi-algebraic geometry (see [27]).

For some examples of known results consider the following. Let R be a real closed field (e.g., the reader may suppose $\mathrm{R} = \mathbb{R}$). Let $P_1, \ldots, P_n \in \mathrm{R}[X_1, \ldots, X_k]$ and suppose that $\deg P_i \leq d$ for all $i = 1, \ldots, n$. Suppose $S \subset \mathrm{R}^k$ is defined by $P_1 = 0, \ldots, P_n = 0$. A classical result in real algebraic geometry, the Oleinik-Petrovsky-Thom-Milnor bound [23, 26, 22], is that the sum of the Betti numbers of S is at most $d(2d-1)^{k-1}$. This result has since been improved many times to extend to a larger class of semi-algebraic sets as well as to bound each of the Betti numbers individually [8, 4, 16].

The results above all provide quantitative bounds on the topological complexity of a semi-algebraic set $S \subset \mathrm{R}^k$ in terms of k and the number and degree of the polynomials in a description of S. However, the degree of a polynomial is only one measure of the complexity of a polynomial, and similar bounds to those above have been shown using other notions of complexity. One such measurement of the complexity of a polynomial which is different from its degree is the additive complexity of a polynomial. Roughly speaking, the additive complexity of a polynomial is the number of additions required in order to evaluate the polynomial at a generic point. This number is generally less than the number of monomials as in the example $\frac{X^{d+1}-1}{X-1}$ which has additive complexity 2 but has d monomials when expanded. Several authors have proved results which bound either the topological complexity or the number of possible topological types of a semi-algebraic set $S \subset \mathrm{R}^k$ in terms of the complexity of the polynomials describing S [9, 11, 14, 27].

2. Research

2.1. Refined bounds on the number of connected components of sign conditions in a variety .

Let $\mathcal{P}$ be a set of s polynomials in k variables. The realization of a sign condition σ in the a semi-algebraic set V is the semi-algebraic set

$$\mathcal{R}(\sigma, V) = \{x \in V | \bigwedge_{P \in \mathcal{P}} \mathrm{sign}(P(x)) = \sigma(P)\}.$$

A sign condition σ is called realizable in V if $\mathcal{R}(\sigma, V)$ is non-empty. In [24], Pollack and Roy proved a bound of $\binom{s}{k} O(d)^k$ on the number of semi-algebraically connected components of realizable sign conditions defined by s polynomials in k variables of degree at most d.

It was soon realised that in certain applications, notably in geometric transversal theory, as well as in bounding the configuration space in robotics, it is useful to study the realizations of sign conditions defined by a family of s polynomials in $\mathrm{R}[X_1, \ldots, X_k]$ restricted to a real variety V where the dimension of V may be much less than k. For example, if all the polynomials involved have degree at most d, it was shown in [7] that the sum of the i-th Betti number over all realizations of realizable sign conditions restricted to a variety of dimension k' is bounded by

$$\sum_{0 \leq j \leq k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1} = \binom{s}{k'} O(d)^k. \tag{1}$$

It has been recently discovered that in some applications in discrete geometry it is necessary to have a further refined bound on the number of connected components of the realizations of sign conditions on a variety where the roles of the degrees of the polynomials defining the variety and the degrees of the polynomials defining the sign condition are distinguished. In particular, Guth and Katz [17] have very recently developed a divide-and-conquer technique based on the Ham Sandwich Theorem as part of their technique for solving the Erdös distinct distance problem, a long standing open question in discrete computational geometry. Their technique has also been very recently used by several authors including Solymosi and Tao [25], Katplan et. al [19], and Zahl [28] to prove new results as well as to provide new and simple proofs of classical results in discrete geometry. However, currently the applications of the Ham Sandwich Theorem to these problems have been restricted to low-dimensional ambient space (typically dimension 2 or 3).

Each of the proofs of these results relies on a bound on the number of connected components of sign conditions in a variety which is similar to special cases (e.g., in low dimension) of the bound in Equation (1) except that there is a more refined dependence on the degrees of the polynomials involved. It was conjectured by Jiri Matousek [20] that a similar bound to Equation (1) with a more refined dependence on the degrees might hold, but the technique of the proof in [7] does not readily extend to a bound with a more refined dependence on the degrees due to a sums of squares approach. We prove the conjecture in our paper [3].

We have proved a refined bound for the general case similar to Equation (1). In particular, we have shown that the number of semi-algebraically connected components of the realizations of all realizable sign conditions is bounded by

$$\sum_{j=0}^{k'} 4^j \binom{s+1}{j} F_{d,d_0,k,k'}(j),$$

where $s = \mathrm{card}\ \mathcal{P}$, and

$$F_{d,d_0,k,k'}(j) = \binom{k+1}{k-k'+j+1} (2d_0)^{k-k'} d^j \ \max\{2d_0, d\}^{k'-j} + 2(k-j+1).$$

This bound can be written more simply as $(sd)^{k'} d_0^{k-k'} O(1)^k$.

It may be possible to use our refined bound on the number of connected components of sign conditions in a variety to prove the existence of an improved ham sandwich decomposition well suited for extending the recent results in discrete geometry to higher dimensions. The difficulties arising in the problem of extending the results to higher dimensions has been discussed by several authors (see [18, 3rd paragraph on page 7][19, Section 3, page 9][28, Section 4, page 12]). Since a result of this kind would be very interesting to both researchers in real algebraic geometry as well as discrete geometry, I am planning to investigate this possibility immediately. Our paper has appeared in *Discrete Comput. Geom.* (DOI: 10.1007/s00454-011-9391-3) and has already been proven useful by several authors considering problems in discrete geometry [19, 25, 28, 30], computational geometry [1], and problems in harmonic analysis related to the Kakeya problem [29].

2.2. **One-parameter limits and additive complexity.** A second type of quantitative results on the topology of semi-algebraic sets is to obtain tight bounds on the number of different topological types of semi-algebraic sets defined by polynomials of fixed complexity. It follows from the well-known Hardt's triviality theorem for o-minimal structures (see [27, 15]) that this number is finite when the complexity of a polynomial is measured by its degree. However, the quantitative bounds that follow from the proof of Hardt's theorem give only doubly exponential bounds on the the number of topological types (unlike the singly exponential bounds on the Betti numbers). Tighter (i.e. singly exponential) bounds have been obtained on the number of of possible *homotopy types* of semi-algebraic sets defined by different classes of formulas of bounded format [9, 5]. Namely, in [9] the authors proved that if $S \subset \mathrm{R}^k$ is a semi-algebraic set described by polynomials whose total number of monomials does not exceed a, then the number of homotopy types of S does not exceed

$$2^{O(ka)^4}.$$

Additionally, several similar results were proved for different notions of complexity of a polynomial. However, the additive complexity case was left open, but it was conjectured that a similar result should hold. The main motivation of our paper is to obtain a singly exponential bound on the number of distinct homotopy types of semi-algebraic defined by polynomials of bounded additive complexity proving the conjecture of [9].

Our proof requires one to consider limits of one-parameter semi-algebraic families of fixed complexity. While the limits of one-parameter semi-algebraic families defined by polynomials with fixed complexity can have complicated descriptions which cannot be described by polynomials of bounded complexity, the topological complexity (for example, measured by their Betti numbers) of such limit sets are well controlled. Indeed, the problem of bounding the Betti numbers of Hausdorff limits of one-parameter families of semi-algebraic sets was considered by Zell in [31], who proved a singly exponential bound on the Betti numbers of such sets. We prove that the number of homotopy types of such limits can indeed be bounded single exponentially in terms of the format of the formulas defining the one-parameter family. The techniques introduced by Zell in [31] (as well certain semi-algebraic constructions described in [10]) play a crucial role in the proof of our bound. These intermediate results may be of independent interest.

In our paper [2], we prove the conjecture of [9] stating that the number of homotopy types of semi-algebraic sets defined by k-variate polynomials of additive complexity bounded by a is bounded by $2^{(k(k+a))^{O(1)}}$. The current project belongs to the setting of real numbers due to the restrictions of the works on which the paper relies. Extending the results of the current paper to a more general setting as well as developing the topological theory of arbitrary real closed fields are potential research areas which I intend to investigate. Our paper is accepted and will appear in *J. Eur. Math. Soc.*

2.3. **Semi-cylindrical decomposition.** We next describe another decomposition technique which has applications in discrete geometry similar to the result described in Section 2.1.

Given $S \subset \mathrm{R}^k$ a semi-algebraic set, a problem is to find a partition of R^k into semi-algebraic subsets which are simple and compatible with S (they either do not intersect S or are contained in S). In this case, the sets are simple if they are described by a constant number of polynomials of fixed maximal degree, with the constant not depending on S. One method to achieve this partition, the Collins method or cylindrical decomposition, is well-known [13]. Quantitative results which give bounds on the number of sets required as well as the degrees of the polynomials which describe them can be found in [8].

There are examples that show that the bound in the quantitative cylindrical decomposition is inherently doubly-exponential, which prevents this method from being computationally viable in many applications. However, the coarsest possible partition of R^k which is compatible with S is that obtained by the sign conditions on the polynomials describing the set S, and this partition requires at most a single-exponential number of cells in the number of polynomials. However, the sets of this partition are not simple in the sense of the previous paragraph in that their description may require all the polynomials in the description of S.

A decomposition of $\mathbb{R}^k$, compatible with S, into single-exponentially many simple sets has been described in the paper [12], but the proof given does not extend to arbitrary real closed fields and thus the result applies only to semi-algebraic subsets of $\mathbb{R}^k$. Results which have been generalized from this context to the theory of o-minimal structures and arbitrary real closed fields is the content of the paper [6], but the analogous result for the semi-cylindrical decomposition was not obtained. We prove that in any o-minimal structure of a real closed field R and for any definable set $S \subset \mathrm{R}^k$, there is a decomposition of R^k into a semi-algebraic family compatible with S such that the number of sets in the partition is singly-exponential in the number of polynomials describing S. Although in applications the improvement of the exponent from doubly-exponential to singly-exponential is desirable, it may be possible to further reduce the exponent. Reducing the exponent from $2k$ to k in the current work would answer a long open question in discrete geometry (see [21]).

3. Ongoing and future research

- The recent work of Guth and Katz [17] has generated interest for many researchers in the area of discrete computational geometry who have proven results using the divide-and-conquer technique referred to in the literature as r-partitioning (cf. Section 2.1). The prototypical example is to prove an upper bound on the number of incidences between m points and n lines in R^k, where a point and line are incident if the line contains the point. Given a finite set of points $X \subset \mathrm{R}^k$, the goal is to find a polynomial $f \in \mathrm{R}[X_1, \ldots, X_k]$ of as low degree as possible such that the connected components of $\mathrm{R}^k - \{f = 0\}$ contain at most $\mathrm{card}(X)/r$ points of X. The so-called Ham Sandwich Theorem asserts the existence of such an f of degree $O(r^{1/k})$. The technique then proceeds by applying a rough bound on each subset $X \cap C$ for C a connected component of $\mathrm{R}^k - \{f = 0\}$ and deal with the points $X \cap \{f = 0\}$ separately using the fact that f has degree $O(r^{1/k})$. However, it may be possible that *all* the points of X lie on the real algebraic set $\{f = 0\}$, and in each of the recent results, dealing with the points on $\{f = 0\}$ seems to be the most difficult case to handle.

One might seek to obtain a partition of R^k into semi-algebraically connected sets of various dimension so that all the points of X are r-separated, that is, for each C in this partition there are at most $\mathrm{card}(X)/r$ points in $X \cap C$ and such that the sets C are described by polynomials of degree $O(r^{1/k})$, but this is impossible. Nevertheless, it may be possible to prove existence of a multi-level decomposition which is well suited to an inductive r-partitioning technique, and although this seems to be a difficult problem it is certainly one for which a positive result would be interesting to researchers in several areas of mathematics.

- On another note, the proof of the refined bound of Section 2.1 does not readily extend to prove existence of an algorithm which computes the number of connected components of sign conditions on a variety with separate inputs for the degrees of the polynomials involved. However, such an algorithm is believed to exist and this constitutes a project on which immediate progress can be made. This result would have an immediate impact in a broad range of applications in areas such as computer aided design and robot motion planning where the best known algorithms have found extensive use. While the result of [3] does not immediately provide such an algorithm, once a quantitative bound of this type is found it is natural to expect that an algorithm with the appropriate complexity can be designed.

- While the refined bound of Section 2.1 pertains only to the number of connected components of sign conditions on a variety, it is believed that this result should extend to a bound on the sum of the Betti-numbers as well. A quantitative result which obtains the same bound as [3] but on the sum of the Betti numbers would improve classical extremely well known theorems in real algebraic geometry. While the proof of the current result does not readily extend to a bound on the higher Betti numbers, we have succeeded in proving an analogous but weaker refined bound on the sum of the Betti-numbers.

References

[1] P. K. Agarwal, J. Matousek, and M. Sharir. On Range Searching with Semialgebraic Sets II. *ArXiv e-prints*, August 2012.

[2] Sal Barone and Saugata Basu. On the homotopy types of limits of semi-algebraic sets and additive complexity of polynomials,. *J. Eur. Math. Soc.*, to appear.

[3] Sal Barone and Saugata Basu. Refined bounds on the number of connected components of sign conditions on a variety. *Discrete Comput. Geom.*, 47:577–597, 2012.

[4] S. Basu. On bounding the Betti numbers and computing the Euler characteristic of semi-algebraic sets. *Discrete Comput. Geom.*, 22(1):1–18, 1999.

[5] S. Basu. On the number of topological types occurring in a parametrized family of arrangements. *Discrete Comput. Geom.*, 40:481–503, 2008.

[6] S. Basu. Combinatorial complexity in o-minimal geometry. *Proc. London Math. Soc. (3)*, 100:405–428, 2010. (an extended abstract appears in the Proceedings of the ACM Symposium on the Theory of Computing, 2007).

[7] S. Basu, R. Pollack, and M.-F. Roy. On the combinatorial and algebraic complexity of quantifier elimination. *J. ACM*, 43(6):1002–1045, 1996.

[8] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2006 (second edition).

[9] S. Basu and N. Vorobjov. On the number of homotopy types of fibres of a definable map. *J. Lond. Math. Soc. (2)*, 76(3):757–776, 2007.

[10] Saugata Basu and Thierry Zell. Polynomial hierarchy, Betti numbers, and a real analogue of Toda's theorem. *Found. Comput. Math.*, 10(4):429–454, 2010.

[11] R. Benedetti and J.-J. Risler. *Real algebraic and semi-algebraic sets.* Actualités Mathématiques. Hermann, Paris, 1990.

[12] B. Chazelle, H. Edelsbrunner, L.J. Guibas, and M. Sharir. A single-exponential stratification scheme for real semi-algebraic varieties and its applications. *Theoretical Computer Science*, 84:77–105, 1991.

[13] G. E. Collins. Quantifier elimination for real closed fields by cylindric algebraic decomposition. In *Second GI Conference on Automata Theory and Formal Languages*, volume 33 of *Lecture Notes in Computer Science*, pages 134–183, Berlin, 1975. Springer- Verlag.

[14] M. Coste. Topological types of fewnomials. In *Singularities Symposium—Lojasiewicz 70 (Kraków, 1996; Warsaw, 1996)*, volume 44 of *Banach Center Publ.*, pages 81–92. Polish Acad. Sci., Warsaw, 1998.

[15] M. Coste. *An introduction to o-minimal geometry*. Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000. Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica.

[16] A. Gabrielov and N. Vorobjov. Betti numbers of semialgebraic sets defined by quantifier-free formulae. *Discrete Comput. Geom.*, 33(3):395–401, 2005.

[17] L. Guth and N. Hawk Katz. On the Erdos distinct distance problem in the plane. *ArXiv e-prints*, November 2010.

[18] Haim Kaplan, Jiří Matoušek, and Micha Sharir. Simple proofs of classical theorems in discrete geometry via the guth-katz polynomial partitioning technique. *Discrete & Computational Geometry*, 48:499–517, 2012. 10.1007/s00454-012-9443-3.

[19] Haim Kaplan, Jiří Matoušek, Zuzana Safernová, and Micha Sharir. Unit distances in three dimensions. *Comb. Probab. Comput.*, 21(4):597–610, July 2012.

[20] J. Matousek. Private correspondence.

[21] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[22] J. Milnor. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.*, 15:275–280, 1964.

[23] I. G. Petrovskiĭ and O. A. Oleĭnik. On the topology of real algebraic surfaces. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 13:389–402, 1949.

[24] R. Pollack and M.-F. Roy. On the number of cells defined by a set of polynomials. *C. R. Acad. Sci. Paris*, 316:573–577, 1993.

[25] Jozsef Solymosi and Terence Tao. An incidence theorem in higher dimensions. *Discrete Comput. Geom.*, 48:255–280, 2012. 10.1007/s00454-012-9420-x.

[26] R. Thom. Sur l'homologie des variétés algébriques réelles. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 255–265. Princeton Univ. Press, Princeton, N.J., 1965.

[27] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.
[28] J. Zahl. An improved bound on the number of point-surface incidences in three dimensions. *ArXiv e-prints*, April 2011.
[29] J. Zahl. On the Wolff circular maximal function. *ArXiv e-prints*, October 2011.
[30] J. Zahl. A Szemeredi-Trotter type theorem in $\mathbb{R}ˆ4$. *ArXiv e-prints*, March 2012.
[31] T. Zell. Topology of definable hausdorff limits. *Discrete Comput. Geom.*, 33:423–443, 2005.