Your name: ____________________________  Section: B1-Renom □  B2-Labate □

part C
type text here

9) 12 pts  = 2 + 10.
   a) If \( x_{n+2} + nx_{n+1} - x_n = n \), \( x_0=2 \), and \( x_1=1 \), find the values of \( x_2 \), \( x_3 \), and \( x_4 \). Show your work.

   Setting \( n=0 \), we obtain \( x_2 + 0x_1 - x_0 = 0 \), so \( x_2=x_0=2 \).
   Setting \( n=1 \), we obtain \( x_3 + x_2 - x_1 = 1 \), so \( x_3=0 \).
   Setting \( n=2 \), we obtain \( x_4 + 2x_3 - x_2 = 2 \), so \( x_4=4 \).

   answer: \( x_2=2 \), \( x_3=0 \), and \( x_4=4 \)

   b) Find a formula for the sum of the first \( n \) squares:

   \[ x_n = 1^2 + 2^2 \ldots + n^2 \]

   Show your work, using the techniques we used for solving linear difference equations with constant coefficients.
   (Note: No credit if you work the problem in a different way, like antidifferences or induction, and no credit just for just giving the answer. State the difference equation you are going to solve, and then solve it using the techniques we learned.)

   \( x_n \) solves the difference equation: \( x_n - x_{n-1} = n^2 \). The characteristic polynomial is \( x-1 \) which has one root: 1. The polynomial for the simplest homogeneous equation that has the righthand side as solution is \( (x-1)^3 \). The product of these two polynomials is \( (x-1)^4 \). The general solution to the homogeneous linear difference equation whose characteristic polynomial is \( (x-1)^4 \) is \( C_11^n + C_2n1^n + C_3n^21^n + C_4n^31^n = C_1 + C_2n + C_3n^2 + C_4n^3 \). From this we delete the solution to the homogeneous case, which is just the first term \( C_1 \). We are therefore led to seek a solution of the form \( x_n = C_2n + C_3n^2 + C_4n^3 \). Or, more simply, we will just write \( x_n = An + Bn^2 + Cn^3 \). We calculate:

   \[ x_n-x_{n-1} = 3Cn^2 + (2B-3C)n + (A-B+C). \]

   So, we want

   \[ 1 = 3C \]
   \[ 0 = 2B - 3C \]
   \[ 0 = A - B + C \]

   which has the unique solution \( C=1/3 \), \( B=1/2 \), \( A=1/6 \). To find the general solution, we add back the homogeneous solution which is \( x_n=D \). The general solution is therefore:

   \[ x_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} + D \quad (D= \text{some constant}). \]

   We can plug in \( n=1 \) to see that the value of \( D \) is zero. Hence

   \[ x_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}. \]
10) 4 pts Describe the smallest set $S$ of simple undirected graphs such that

i. The graph consisting of just a single node (and no edges) belongs to $S$.

ii. If $G$ belongs to $S$, $i$ is a node of $G$, and $i'$ is not a node of $G$, then the graph obtained by adding the node $i'$ and the arc $(i, i')$ also belongs to $S$.

the set of trees containing one or more nodes. These answers will also be accepted: the set of all trees, the set of all rooted trees.

11) 12 pts = 4 pts each. In the following counting problems, express the answer as an expression (such as $3 \cdot 5! - 4!$) and then also as an exact numerical value (such as 336). To get credit, the expression (eg. $3 \cdot 5! - 4!$) must be correct, whereas if you evaluate the formula wrongly, the penalty will be just one point.

a) In how many ways can six hats be given to six people if at least one person gets her own hat? (Assume that exactly one hat belongs to each person). Hint: use the inclusion-exclusion principle.

answer: $6 \cdot 5! - C(6;2)4! + C(6;3)3! - C(6;4)2! + C(6;5)1! - C(6;6)0! = 455$

remark: If you count the number of ways to give six hats so nobody gets their own hat, then that would be the number of derangements of six objects, a topic we studied in class and in homework exercises. Subtract that number from $6!$ and you have the answer to this problem.

b) From a group of 12 people, in how many ways can three people be awarded a chocolate ice cream cone, three people be awarded a vanilla ice cream cone, three people be awarded a coffee ice cream cone (and three people get no ice cream cone)? No person can receive more than one ice cream cone.

\[
\frac{12!}{3!3!3!3!} = 369,600
\]

alternatively, you could compute $C(12,3) \cdot C(9,3) \cdot C(6,3) \cdot C(3,3)$. 

c) In how many ways can 8 people stand in line for a photograph if four of them (Bob, Mary, Joanne, and Rosemary) stand together?

answer: $5 \cdot 4! \cdot 4! = 2880$