stochastic matrices

are square matrices in which every column sums to one.

rental car example (p. 693)

how it can be given a probabilistic interpretation, rather than a deterministic one

markov chains

A markov chain has finitely many states 1,...,k.

\( p_{ij} \) = probability of transition from state \( j \) to \( i \)

The matrix \( P = (p_{ij}) \) is a stochastic matrix, called the transition matrix for the markov chain.

We would turn the rental car example into a Markov chain with the transition matrix \( P = \)

\[
\begin{pmatrix}
NY & LA \\
NY & 1/2 & 1/3 \\
LA & 1/2 & 2/3
\end{pmatrix}
\]

The \( ij \) entry of the power \( P^n \) gives the probability of transition from state \( j \) to state \( i \) in \( n \) steps. (See Theorem 2 on p. 711.) For example, in the rental car example, \( P^2 = \begin{pmatrix} 5/12 & 7/18 \\ 7/12 & 11/18 \end{pmatrix} \) and \( P^3 = \begin{pmatrix} 29/72 & 43/108 \\ 43/72 & 65/108 \end{pmatrix} \), so if the current state is "LA" then the probability that the state is "NY" after three months is 43/108. Here is the tenth power \( P^{10} \):

\[
\begin{pmatrix}
0.4000000000000000 & 0.3999999999999999 \\
0.5000000000000000 & 0.5000000000000000 \\
0.2499999999999999 & 0.2500000000000000 \\
0.2500000000000000 & 0.2499999999999999
\end{pmatrix}
\]

In the long run, what percentage of its time does a car spend in LA? In NY?

HW: p.716 (section 8.10): 1, 2, 3, 5, 8

#2: the fourth power of the transition matrix is

\[
\begin{pmatrix}
1 & 0 & 61/216 & 305/1296 \\
0 & 1 & 305/1296 & 61/216 \\
0 & 0 & 625/1296 & 0 \\
0 & 0 & 0 & 625/1296
\end{pmatrix}
\]

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Why \( P^2 \) gives the transition probabilities for a two-step process

example 2 (p.708): crossings with hybrids.  The transition matrix is

\[
M = \begin{pmatrix}
1/2 & 1/4 & 0 \\
1/2 & 1/4 & 1/2 \\
0 & 1/4 & 1/2
\end{pmatrix}
\]

Here is the 20th power of the transition matrix.  What is it telling us?

\[
\begin{pmatrix}
0.25000004768371582 & 0.25 & 0.2499995231628418 \\
0.5 & 0.5 & 0.5 \\
0.2499995231628418 & 0.25 & 0.2500004768371582
\end{pmatrix}
\]

One is an eigenvalue, and each column of the matrix is an eigenvector for the eigenvalue 1 representing the "steady state".
Knowing that one is an eigenvalue, we can find the steady state by reducing the matrix $M-I$. In fact, $M-I$ reduces to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$.

example 3 (p. 709): toss a die until 6 shows up. The transition matrix is

- A has won: $\begin{pmatrix} 1 & 0 & 1/6 & 0 \\ 0 & 1 & 0 & 1/6 \\ 0 & 0 & 0 & 5/6 \\ 0 & 0 & 5/6 & 0 \end{pmatrix}$
- A's turn: $\begin{pmatrix} 0 & 0 & 0 & 5/6 \\ 0 & 0 & 5/6 & 0 \end{pmatrix}$
- B's turn: $\begin{pmatrix} 0 & 0 & 0 & 5/6 \\ 0 & 0 & 5/6 & 0 \end{pmatrix}$

Here is the $100^{th}$ power of the transition matrix. What is it telling us?

$\begin{pmatrix} 1.0 & 0.0 & 0.5454545388683602 & 0.45454544905696675 \\ 0.0 & 1.0 & 0.45454544905696675 & 0.5454545388683602 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$

It seems to be converging to

$\begin{pmatrix} 1.0 & 0.0 & 6/11 & 5/11 \\ 0 & 1.0 & 5/11 & 6/11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Notice that there is a slight advantage to being the first player to toss (as one would expect).

Absorbing state

Notice that the only really interesting part of this matrix is the submatrix $\begin{pmatrix} 6/11 & 5/11 \\ 5/11 & 6/11 \end{pmatrix}$. Theorem 4 (p. 714) gives us a direct way to compute this. For a Markov chain with absorbing states whose transition matrix is $\begin{pmatrix} I & R \\ 0 & Q \end{pmatrix}$, the probabilities for eventual transfer from each transient state to each absorbing state are given by the matrix $R(I-Q)^{-1}$. In our example (see transition matrix on p.709),

$R=\begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}$ and $Q=\begin{pmatrix} 0 & 5/6 \\ 5/6 & 0 \end{pmatrix}$, so $R(I-Q)^{-1}=\frac{1}{6} \begin{pmatrix} 1 & -5/6 \\ -5/6 & 1 \end{pmatrix}^{-1}$

$=\frac{1}{6} \begin{pmatrix} 36 & 5/6 \\ 5/6 & 1 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}$

regular markov chains

definition: there exists $n$ such that from any state you can get to any other in $n$ steps.

This is the same as saying that some power of the transition matrix has no zero entries.

$\lambda=1$ is an eigenvalue of the transition matrix $A$ and the corresponding eigenvector is a probability vector $w$ representing the steady state. For any initial probability vector $p$, the sequence $A^np$ converges to $w$.

homework: p.717: 13

What is the steady state for this Markov chain? $\begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/4 & 1/3 & 1/4 \\ 1/4 & 1/3 & 1/2 \end{pmatrix}$? (answer: $\begin{pmatrix} 4/11, 3/11, 4/11 \end{pmatrix}$.

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What is the steady state for this Markov chain?

\[
\begin{pmatrix}
  1/2 & 0 & 0 & 1 \\
  1/2 & 0 & 0 & 0 \\
   0 & 1/2 & 0 & 0 \\
   0 & 1/2 & 1 & 0 \\
\end{pmatrix}
\]

Also draw a state diagram and see if you can see that it is regular.  (answer: (4/9,2/9,1/9,2/9).  The matrix P-I reduces to ((1 0 0 -2) (0 1 0 -1) (0 0 1 -1/2) (0 0 0 0)).

notice in 13b that it always takes an odd number of steps to go from 2 to 3 but an even number of steps to go from 2 to 4.

Note: There is a row reduction "machine" located at the following url: