March 22, 2000

**elementary row operations**
1. multiply a row by a nonzero constant
2. switch two rows
3. add any multiple of one row into another row

**reduced matrices**
A matrix is reduced if it has these properties:
1. In every nonzero row, the first nonzero entry is a one (which we refer to as a leading one.
2. The leading ones progress to the right
3. Each leading one is the only nonzero entry in its column

**solving systems of linear equations**
There are three possible outcomes

hw: p. 621: 3, 5, 11, 12

go to http://mathphysics.com/spingarn/vec/index.html and do these exercise sets:
linear systems and matrices
  row reduction (3)
    1. reducing a 3 x 5 matrix
    2. reducing a 4 x 3 matrix
    3. row equivalence
systems of linear equations (5)
  1. solving a system of three linear equations in four unknowns
  2. finding specific solutions to a linear system
  3. solving a system of four linear equations in four unknowns
  4. a homogeneous system of four equations in five unknowns
  5. a system of four equations in four unknowns

go to http://mathphysics.com/spingarn/lane/index.html and read these files
  elementary row operations
  describing the solution set of a linear system
  Gauss-Jordan method

**an example**
To solve the system
\[
\begin{align*}
2x_1 - x_2 + 3x_4 &+ x_5 = 13 \\
x_1 + x_2 + 6x_3 + x_4 + 8x_5 &= 7 \\
-x_1 + x_2 + 2x_3 &+ 2x_5 = -3 \\
x_1 &+ 2x_3 + x_4 + 3x_5 = 6
\end{align*}
\]
you would reduce the matrix
\[
\begin{pmatrix}
2 & -1 & 0 & 3 & 1 & 13
\end{pmatrix}
\]
Math 2602, Winter/Spring 2000

\[
\begin{array}{cccccc}
1 & 1 & 6 & 1 & 8 & 7 \\
-1 & 1 & 2 & 0 & 2 & -3 \\
1 & 0 & 2 & 1 & 3 & 6 \\
\end{array}
\]

to obtain the reduced matrix:

\[
\begin{array}{cccccc}
1 & 0 & 2 & 0 & 3 & 4 \\
0 & 1 & 4 & 0 & 5 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
\end{array}
\]

From the reduced matrix, you can read the solution to the system:

\[
\begin{align*}
x_1 &= 4 - 2x_3 - 3x_5 \\
x_2 &= 1 - 4x_3 - 5x_5 \\
x_4 &= 2 \\
x_3 &= \text{any number} \\
x_5 &= \text{any number}
\end{align*}
\]

The solution can also be written like this:

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 
\end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ x_3, x_5 \in \mathbb{R}.
\]

p.637: 1,2,3,5a-g,8

The row reductions required for these problems are as follows:

\[
\begin{array}{cccc}
1 & 1 & 1 & 0 \\
-1 & 1 & 3 & -2 \\
2 & -1 & 4 & 4 \\
\end{array}
\]

reduces to

\[
\begin{array}{cccc}
1 & 0 & 0 & 9/8 \\
0 & 1 & 0 & -5/4 \\
0 & 0 & 1 & 1/8 \\
\end{array}
\]

"*******************"

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & -1 & 3 \\
\end{array}
\]

reduces to

\[
\begin{array}{cccc}
1 & 0 & 3 & -1 \\
0 & 1 & -2 & 2 \\
\end{array}
\]

"*******************"

\[
\begin{array}{cccc}
2 & -2 & 3 & 1 \\
-1 & 1 & 0 & 1 \\
3 & -3 & 5 & 2 \\
\end{array}
\]

reduces to

\[
\begin{array}{cccc}
1 & -1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

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reduces to
1 0 -1 -2
0 1 2 3

reduces to
1 0 0 0
0 1 3/2 0
0 0 0 1

reduces to
1 0 0 0
0 1 3/2 0
0 0 0 1

reduces to
0 0 1
0 0 1
0 0 0

reduces to
1 3 0 1 0
0 0 1 1/3 0
0 0 0 0 1

reduces to
1 2 2
2 1 -2
2 -2 1

reduces to
1 0 0
0 1 0
0 0 1
March 27, 2000

LU-decomposition

Here is another example, similar to the one we did in class. Suppose we want to solve systems of linear equations of the form

\[
\begin{align*}
3x + 5y + z &= b_1 \\
6x + y - 5z &= b_2 \\
-2x + 4y &= b_3
\end{align*}
\]

The first step is to reduce the coefficient matrix to upper triangular form:

\[
\begin{pmatrix}
3 & 5 & 1 \\
6 & 1 & -5 \\
-2 & 4 & 0
\end{pmatrix}
\]

partially reduces to

\[
\begin{pmatrix}
3 & 5 & 1 \\
0 & -9 & -7 \\
0 & 0 & -136/27
\end{pmatrix}
\]

by performing this sequence of elementary operations:

1. subtract \(2\times\) row 1 from row 2
2. subtract \(-2/3\times\) row 1 from row 3
3. subtract \(-22/27\times\) row 2 from row 3

We store these numbers in the positions where the zeros are located in the partially reduced matrix:

\[
\begin{pmatrix}
3 & 5 & 1 \\
2 & -9 & -7 \\
-2/3 & -22/27 & -136/27
\end{pmatrix}
\]

The numbers in this matrix give us all the information we need to efficiently solve the system (*) for any choice of \(b_1, b_2, b_3\).

For example, suppose we want to solve

\[
\begin{align*}
3x + 5y + z &= 6 \\
6x + y - 5z &= 23 \\
-2x + 4y &= -18
\end{align*}
\]

Starting with the vector \(\begin{pmatrix} 6 \\ 23 \\ -18 \end{pmatrix}\), we perform nine steps, one for each entry in the LU decomposition:

1. subtract \(2\times\) row 1 from row 2 to obtain \(\begin{pmatrix} 6 \\ 11 \\ -18 \end{pmatrix}\)
2. subtract \(-2/3\times\) row 1 from row 3 to obtain \(\begin{pmatrix} 6 \\ 11 \\ -14 \end{pmatrix}\)
3. subtract \(-22/27\times\) row 2 from row 3 to obtain \(\begin{pmatrix} 6 \\ 11 \\ -136/27 \end{pmatrix}\)
4. divide row 3 by \(-136/27\) to obtain \(\begin{pmatrix} 6 \\ 11 \\ 1 \end{pmatrix}\)
5. subtract \(-7\times\) row 3 from row 2 to obtain \(\begin{pmatrix} 6 \\ 18 \\ 1 \end{pmatrix}\)
6. subtract 1× row 3 from row 1 to obtain \[
\begin{pmatrix}
5 \\
18 \\
1
\end{pmatrix}
\]

7. divide row 2 by -9 to obtain \[
\begin{pmatrix}
5 \\
-2 \\
1
\end{pmatrix}
\]

8. subtract 5× row 2 from row 1 to obtain \[
\begin{pmatrix}
15 \\
-2 \\
1
\end{pmatrix}
\]

9. divide row 1 by 3 to obtain \[
\begin{pmatrix}
5 \\
-2 \\
1
\end{pmatrix}
\]

In this way we obtain the solution \(x=5,\ y=-2,\ z=1\).

homework: p. 653: 3, 6, 7

matrix multiplication

Today, we also talked about the definition of matrix multiplication, and how to view matrix multiplication in terms of taking linear combinations of rows.

March 29, 2000

p. 663, section 8.6: 2, 3, 4, 9, 10, 11, 21, 23, 24, 36