February 9, 2000

Today we worked several of the above problems, including 17, 19, 21, 23, 24, 26, 57, 28, 29.

Then we discussed the theorems in section 4.5.

**homework:** p. 314 (section 4.5): 1,2,3a,4,5

1. All three expressions count the number of ways of choosing k+1 things from n and labelling one of them as special:
   a) \((n-k) \binom{n}{k}\) says pick k things and then pick one of the remaining n-k to be the special one.
   b) \((k+1) \binom{n}{k+1}\) says pick k+1 things and then pick one of the ones already picked to be the special one.
   c) \(n \binom{n-1}{k}\) says pick the special one first, and then pick the k unspecial ones from the remaining n-1.

2. You can write this as \((k+1) \binom{n+1}{k+1}\) = \((n+1) \binom{n}{k}\) so it is the same formula that was proved in #1 (just replace n-1 with n in #1).

3. Here are the counting arguments:
   a) This is the number of ways to pick, from a set of n students, k to place in dormitory A, and j to place in dormitory B. It doesn't matter whether you first pick the ones for dormitory A or B.
   b) This is the same as part a, but the second expression says first pick the k+j students who will be placed in either dormitory, decide which k of those go into dormitory A, and put the rest in dormitory B.
   c) Both expressions count the number of ways to pick, from a set of n things, an unordered subset of k things and a disjoint ordered subset of j things. It doesn't matter which set you pick first.

4. A subset of k things is to be picked from n things. Label two of them as a and b. There are \(\binom{n-2}{k}\) ways to pick the k things if a and b are not used. There are \(\binom{n-2}{k-2}\) ways to pick the k things if a and b are both used. There are \(\binom{n-2}{k-1}\) ways to pick the k things if a but not b is used, and there are \(\binom{n-2}{k-1}\) ways to pick the k things if b but not a is used.

5. \(P(n,k)\) is the number of ways to pick k things in order from n. This is the same as the number of picking the first thing times the number of picking the other k-1 in order.

February 11, 2000

Next test is Tuesday March 29.

Today we discussed the binomial theorem and the multinomial theorem.

**homework:** p. 323: 1b, 2b, 3, 4be, 5, 7a, 15, 16, 17

7. For a set of size n, the number of subsets with an odd number of elements
is equal to the number of subsets with an even number of elements. This tells you that $C(n,0)+C(n,2)+C(n,4)+\ldots$ counts half the total number of subsets, which is $2^{n-1}$.

15. 30 and 60

16. Yes, they are the same. Both count the number of ways to divide 6 things into three categories A,B,C, putting 1 into category A, 2 into category B, and 3 into category C.

17. a) $(1+1+1)^n = 3^n$
   
   b) $(1/3 + 2/3 -1)^n = 0$

February 14, 2000

review

$$C(n;k) = \binom{n}{k}. \quad P(n;k).$$

Think of a question for which the answer is:

$$\binom{15}{4}, P(15,4), \binom{15}{4} \binom{11}{3}, 15!, 15!4!, \frac{17!}{3!12!2!}, 2^{15}\cdot 15!, \binom{17}{3}^5,$$

$$2^{15}\cdot 15!, \binom{17}{3}^5.$$

$$\binom{15}{4}^4!$$ is the same as what number. Show this by counting the same thing in two different ways.

inclusion-exclusion principle (section 4.8)

How many numbers in 1,...,100 have no factor of 3,5, 7,11?

Writing $10+3=3$, for example, to write integer division (throwing away the remainder), the answer is

$$100+3 + 100+5 + 100+7 + 100+11$$

$$- 100+15 - 100+21 - 100+33 - 100+35 - 100+55 - 100+77$$

$$+ 100+(3\cdot 5\cdot 7) + 100+(3\cdot 5\cdot 11) + 100+(3\cdot 7\cdot 11) + 100+(5\cdot 7\cdot 11)$$

$$- 100+(3\cdot 5\cdot 7\cdot 11)$$

$$= 33 + 20 + 14 + 9$$

$$- 6 - 4 - 3 - 2 - 1 - 1$$

$$+ 0 + 0 + 0 + 0$$

$$- 0$$

$$= 59$$

So, 100-59=41 DO have a factor of 3,5,7, or 11.

We worked examples 3 and 4 in section 4.8

HOMEWORK: p. 341: 1,5,7,11,12,13,14  (we did 7 and 11 in class)

solutions:

5. "relatively prime to 105" means the same as "not divisible by 3, 5, or 7". Let

A = the integers between 1 and 105 that are divisible by 3

B = the integers between 1 and 105 that are divisible by 5

C = the integers between 1 and 105 that are divisible by 7

If we can find $|A\cup B\cup C|$ then the answer to #5 is 105$-|A\cup B\cup C|$. By the inclusion-exclusion principle,

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C| + |A\cap B\cap C|$$

$$= 35 + 21 + 15 - 7 - 5 - 3 + 1 = 57$$
so \( 105 - |A \cup B \cup C| = 105 - 57 = 48 \).

7. The question is how many of the integers 1,...,99 do not have 4,9,or 25 as a factor?
The answer is the same if we replace 99 with 100:
How many integers 1,...,100 do not have 4,9,or 25 as a factor?
The answer is the same as 100-(the number of integers 1..100 that DO HAVE 4,9,or 25 as a common factor.
This is
\[
100 - \text{(\#that have 4 as a factor)} - \text{(\#that have 9 as a factor)} - \text{(\#that have 25 as a factor)} + \text{(\#that have 4 and 9 as factors)} + \text{(\#that have 4 and 25 as factors)} + \text{(\#that have 9 and 25 as factors)} - \text{(\#that have 4,9,and 25 as factors)}
\]
\[
= 100 - 25 - 11 - 4 + 2 + 1 + 0 - 0 = 63
\]

11. Let \( A_i \) be the set of of permutations of 1..n that do keep i in position
1. Notice that
\[
|A_i| = (n-1)!
\]
if \( i \neq j \) then \( |A_i \cap A_j| = (n-2)! \)
if \( i,j,k \) are distinct, then \( |A_i \cap A_j \cap A_k| = (n-3)! \)
... and so forth.
So the number of permutations that leave some i in position is
\[
|A_1 \cup A_2 \cup A_3 \cup ...| = |A_1| + |A_2| + \text{all singles} - |A_1 \cap A_2| - \text{all pairs} + |A_1 \cap A_2 \cap A_3| + \text{all triples} - \text{... and so forth}
\]
\[
= \binom{n}{1} (n-1)! - \binom{n}{2} (n-2)! + \binom{n}{3} (n-3)! - \ldots
\]

12. There are 11 letters. Treat the two a's as if they were just one letter. Then there would be just 10 letters and 10! possible permutations. However the two a's can be permuted in 2 ways, so there are \( 2 \cdot 10! \) ways to permute the 11 letters, keeping the two a's together. There would be \( 11! - 2 \cdot 10! = 9 \cdot 10! \) permutations that do not have this property.

13. It is easier to count the number of permutations in which two identical letters are next to each other. Let
\begin{align*}
A &= \text{the set of permutations in which the a's are together} \\
B &= \text{the set of permutations in which the b's are together} \\
C &= \text{the set of permutations in which the c's are together}
\end{align*}
By the inclusion-exclusion principle,
\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]
\[
= 2 \cdot 5! + 2 \cdot 5! + 2 \cdot 5! - 2 \cdot 2 \cdot 4! - 2 \cdot 2 \cdot 4! - 2 \cdot 2 \cdot 4! + 2 \cdot 2 \cdot 2 \cdot 3! = 480
\]
So the number of permutations in which no two identical letters are together is \( 6! \cdot 480 = 240 \).

Here is another way to do #13 -- but be careful, it is far easier to make a mistake if you do it this way: Break it up into two cases -- either the terms in positions 1 and 3 are the same or they are different. The number of permutations of aabbcc in which terms 1 and 3 are the same and
in which no consecutive terms are alike is
\[ 6 \cdot 4 \cdot 1 \cdot 2 \cdot 1 \cdot 1 = 48 \]
(because 6 can go in the first spot, 4 in the second spot, and so forth).
The number of permutations of aabbcc in which terms 1 and 3 are different
and in which no consecutive terms are alike is
\[ 6 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 192 \]
The total number of permutations of aabbcc in which no consecutive terms
are alike is 48 + 192 = 240.