January 24, 2000

possible outcomes (p. 60)

<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>optimal</th>
<th>infeasible</th>
<th>unbounded</th>
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<tr>
<td>optimal</td>
<td>possible</td>
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<tr>
<td>DUAL</td>
<td>infeasible</td>
<td>impossible</td>
<td>possible</td>
</tr>
<tr>
<td>unbounded</td>
<td>impossible</td>
<td>possible</td>
<td>impossible</td>
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</tbody>
</table>

Remarks: if one problem has a solution, so does the other. Furthermore the optimal values are equal (see theorem 5.1 on page 58).

Both problems can be infeasible. In that case, you could say that \( \sup (P) = -\infty \) and \( \inf (D) = +\infty \). This is the only case where the optimal values can be different.

Example: \((P)\) max \(2x_1 - x_2\) subject to \(x_1 - x_2 \leq 1\), \(-x_1 + x_2 \leq -2\), \(x_1 \geq 0\), \(x_2 \geq 0\) is infeasible. It's dual is also infeasible.

Otherwise, infeasibility is paired with unboundedness:

- If \((P)\) is unbounded, then \((D)\) is infeasible and \(\sup (P) = \inf (D) = +\infty\).
- If \((D)\) is unbounded then \((P)\) is infeasible and \(\sup (P) = \inf (D) = -\infty\).

Note: A supremum over an empty set is considered to be \(-\infty\) and an infimum over an empty set is considered to be \(+\infty\).

Why should this be so? Consider these rules:

\[
\inf (A \cup B) = \min \{\inf (A), \inf (B)\}
\]

and

\[
\sup (A \cup B) = \max \{\sup (A), \sup (B)\}.
\]

How must \(\inf \emptyset\) and \(\sup \emptyset\) be defined in order that these rules continue to hold when the sets \(A\) and/or \(B\) are empty?

The dual of the dual is the primal

To see this, first write \((D)\) in standard form:

\[
\max -(b_1 y_1 + \ldots + b_m y_m) \text{ subject to } \quad -(a_{11} y_1 + \ldots + a_{m1} y_m) \leq -c_1 \\
\quad \ldots \\
\quad -(a_{1n} y_1 + \ldots + a_{mn} y_m) \leq -c_n \\
\text{all } y_i \geq 0.
\]

Treating this as if it were the primal, write down its dual:

\[
\min -(c_1 x_1 + \ldots + c_n x_n) \text{ subject to } \quad -(a_{11} x_1 + \ldots + a_{1n} x_n) \geq -b_1 \\
\quad \ldots
\]
\[
-(a_{m1}x_1 + \ldots + a_{mn}x_n) \geq -b_m \\
\text{all } x_j \geq 0
\]

and you see that this problem is equivalent to (P).

**Motivation for the Dual**

An example: (P) \(\text{max } 3x_1 + 2x_2 \) subject to \(x_1 \geq 0 \) and \(x_2 \geq 0\)

\[
\begin{align*}
-x_1 + 2x_2 & \leq 4 \\
x_1 + x_2 & \leq 8 \\
2x_1 + x_2 & \leq 13 \\
x_1 - x_2 & \leq 5
\end{align*}
\]

Every feasible solution provides a lower bound for the optimal value. For example, \((5,1)\) is feasible, so \(3(5)+2(1) = 17 \leq \text{max}(P)\).

There is also a way to find upper bounds. Suppose we have found four nonnegative numbers \(y_1, y_2, y_3, y_4\) such that

\[
\begin{align*}
-y_1 + y_2 + 2y_3 + y_4 & \geq 3 \\
2y_1 + y_2 + y_3 - y_4 & \geq 2
\end{align*}
\]

of (D).

Then if \(x=(x_1,x_2)\) is feasible for (P), we can conclude that

\[
\begin{align*}
3x_1 + 2x_2 & \leq (-y_1 + y_2 + 2y_3 + y_4)x_1 + (2y_1 + y_2 + y_3 - y_4)x_2 \quad \text{(because } x_1, x_2 \geq 0) \\
& = y_1(-x_1 + 2x_2) + y_2(x_1 + x_2) + y_3(2x_1 + x_2) + y_4(x_1 - x_2) \\
& \leq 4y_1 + 8y_2 + 13y_3 + 5y_4 \quad \text{(because } x \text{ feasible and } y_1 \geq 0)
\end{align*}
\]

so \(4y_1 + 8y_2 + 13y_3 + 5y_4\) is an upper bound for (P). In other words, every feasible point for (D) gives an upper bound for (P).

**Examples**

\(y=(1,1,1,1)\) is feasible for D, so 30 is an upper bound for (P).
\(y=(0,1,1,0)\) is feasible for D, so 21 is an upper bound for (P).

The dual (D) is the problem of finding the smallest such upper bound for (P).

\(c^Tx \leq b^Ty\) whenever \(x\) is feasible for (P) and \(y\) is feasible for (D)

**Proof:**

\[
c^Tx = c_1x_1 + \ldots + c_nx_n \leq (a_{11}y_1 + \ldots + a_{m1}y_m)x_1 + \ldots + (a_{1n}y_1 + \ldots + a_{mn}y_m)x_n
\]

\[
= (a_{11}x_1 + \ldots + a_{1n}x_n)y_1 + \ldots + (a_{m1}x_1 + \ldots + a_{mn}x_n)y_m
\]

\[
\leq b_1y_1 + \ldots + b_my_m = b^Ty.
\]

Read this carefully and try to see where each constraint is used.

**Matrix Formulation**

**PRIMAL**

\[
\begin{align*}
\text{max } c^Tx \\
\text{subject to } Ax & \leq b, \ x \geq 0
\end{align*}
\]

**DUAL**

\[
\begin{align*}
\text{min } b^Ty \\
\text{subject to } A^Ty & \geq c, \ y \geq 0
\end{align*}
\]

The inequality is established as follows. Assume \(Ax \leq b, \ x \geq 0, \ A^Ty \geq c, \ y \geq 0\).

Then:
If $x$ is feasible for $(P)$ and $y$ is feasible for $(D)$, we have seen that $c^t x \leq b^t y$.

Consequently, $c^t x \leq \sup(P) \leq \inf(D) \leq b^t y$.

Also, if $x$ is feasible for $(P)$ and $y$ is feasible for $(D)$, and $c^t x = b^t y$, then $x$ solves $(P)$ and $y$ solves $(D)$.

"Certificate of optimality"

Example: In the example considered above, $(5,3)$ is feasible for $(P)$ and $(0,1,1,0)$ is feasible for $(D)$. Since $c^t x = 3(5) + 2(3) = 21$ and $b^t y = 4(0) + 8(1) + 13(1) + 5(0) = 21$, we know that $(5,3)$ is a maximizer for $(P)$, $(0,1,1,0)$ is a minimizer for $(D)$, and that the optimal value for both problems is 21.

HW#13 Consider the following linear programming problem (note that it is the same as the one in HW#9):

- to maximize $z = x + 2y$ subject to the constraints
  - $x \geq 0$, $y \geq 0$, $y \leq 2$, $x + y \leq 4$, $y - x \leq 1$
  a) Write down the dual.
  b) Find a solution (the point where the optimum is achieved) to the dual.
  c) Display a "certificate of optimality" for the primal and dual solutions.

HW#14: Consider the linear programming problem

(P) maximize $x_1 + 2x_2$ subject to the constraints

- $x_1 + 5x_2 \leq 7$
- $3x_1 + 2x_2 \leq 8$
- $x_1 + x_2 \leq 2$
- $x_1 \geq 0$, $x_2 \geq 0$

Explain why the optimal value in $(P)$ can be no greater than 3.25. In your argument, make no explicit mention of the dual problem and do not compute the optimal value of $(P)$. Pretend you have never heard about the dual. However, you may find it helpful to secretly solve $(P)$ and the dual to help you develop your argument.