ASYMPTOTICS OF THE COLORED JONES FUNCTION OF A KNOT

STAVROS GAROUFALIDIS AND THANG TQ LÊ

Dedicated to Louis Kauffman on the occasion of his 60th birthday

Abstract. To a knot in 3-space, one can associate a sequence of Laurent polynomials, whose nth term is the nth colored Jones polynomial. The paper is concerned with the asymptotic behavior of the value of the nth colored Jones polynomial at $e^{\alpha/n}$, when α is a fixed complex number and n tends to infinity. We analyze this asymptotic behavior to all orders in $1/n$ when α is a sufficiently small complex number. In addition, we give upper bounds for the coefficients and degree of the nth colored Jones polynomial, with applications to upper bounds in the Generalized Volume Conjecture. Work of Agol-Dunfield-Storm-W.Thurston implies that our bounds are asymptotically optimal. Moreover, we give results for the Generalized Volume Conjecture when α is near $2\pi i$. Our proofs use crucially the cyclotomic expansion of the colored Jones function, due to Habiro.

Contents

1. Introduction 2
1.1. Asymptotics of the colored Jones function of a knot 2
1.2. The generalized volume conjecture 4
1.3. The Generalized Volume Conjecture near $2\pi i$ 5
1.4. Upper bounds for the generalized volume conjecture 5
1.5. Relation with hyperbolic geometry, and optimal bounds 5
1.6. The main ideas and organization of the paper 7
1.7. Acknowledgement 7
2. Weak versus strong convergence 8
2.1. A lemma from complex analysis 8
2.2. The main difficulty for uniform bounds 8
3. Two expansions of the colored Jones polynomial 8
3.1. The loop expansion 8
3.2. The cyclotomic expansion 9
3.3. Comparing the cyclotomic and the loop expansion 10
4. A reduction of Theorem 10 to estimates of the cyclotomic function 10
4.1. Uniform bounds of the colored Jones function 10
4.2. Proof of Theorem 10, assuming Theorem 11 11
5. A reduction of Theorem 4 to estimates of the cyclotomic function 12
5.1. Some estimates 12
5.2. Proof of Theorem 4 assuming Theorem 11 13
6. Bounds for the degree and coefficients of the colored Jones function 14
6.1. Bounds for the degree 14
6.2. Bounds for the coefficients 14

Date: November 18, 2006.
The authors were supported in part by National Science Foundation.

1991 *Mathematics Classification.* Primary 57N10. Secondary 57M25.
Key words and phrases: hyperbolic volume conjecture, colored Jones function, Jones polynomial, R-matrices, regular ideal octahedron, weave, hyperbolic geometry, Catalan's constant, Borromean rings, cyclotomic expansion, loop expansion, asymptotic expansion, WKB, q-difference equations, asymptotics, perturbation theory, Kontsevich integral.

7. Bounds for the degree and coefficients of the cyclotomic function 16
7.1. The bound for degrees of $H_{K,n}$ 16
7.2. The bound for the l^1-norm of $H_{K,n}$ 17
8. Growth rates of R-matrices and the Lobachevsky function 18
8.1. The Lobachevsky function 18
8.2. Asymptotics of the R-matrix using ideal octahedra 19
8.3. The maximum of the growth rate of the R-matrix 20
8.4. The maximum of the R-matrices at roots of unity 22
8.5. Proof of Theorem 9 22
9. The generalized volume conjecture near $\alpha = 2\pi i$ 22
10. The q-holonomic point of view 23
10.1. Bounds on l^1-norm of q-holonomic functions 23
10.2. Bounds for higher rank groups 24
11. Some physics 24
11.1. A small dose of physics 24
11.2. The WKB method 25
Appendix A. The volume conjecture for the Borromean rings 25
Appendix B. The volume conjecture for torus knots 27
Appendix C. Proof of Proposition 8.8 27
References 29

1. Introduction

1.1. **Asymptotics of the colored Jones function of a knot.** To a knot K in 3-space, one can associate a sequence of Laurent polynomials

$$J_{K,n}(q) \in \mathbb{Z}[q^{\pm}]$$

for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. $J_{K,2}(q)$ is the famous *Jones polynomial* of K introduced by Jones in [J], and $J_{K,n}(q)$ are roughly speaking the Jones polynomials of $(n-1)$-parallels of the knot. More precisely, $J_{K,n}(q)$ is the *quantum group* invariant of K using the n-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation, normalized by $J_{\mathrm{unknot},n}(q) = 1$ for all n; see [RT, Tu]. The sequence $\{J_{K,n}(q)\}$ for $n \in \mathbb{N}$ is often called the *colored Jones function* of the knot K.

The paper is concerned with the asymptotic growth of the colored Jones function. More precisely, fix a knot K and consider the sequence of holomorphic functions:

$$f_{K,n} : \mathbb{C} \longrightarrow \mathbb{C}, \quad f_{K,n}(z) := J_{K,n}(e^{z/n})$$

for $n \in \mathbb{N}$. In other words, we are evaluating the n-th polynomial $J_{K,n}(q)$ at a complex n-th root of e^z. We will be concerned with strong and weak convergence of the sequence $f_{K,n}$, for $n \in \mathbb{N}$. Let us explain what we mean by that. Fix an open subset U of $\mathbb{C}$ containing 0.

Definition 1.1. (a) A sequence of holomorphic functions $f_n : U \longrightarrow \mathbb{C}$ *strongly converges in* U to a holomorphic function $f : U \longrightarrow \mathbb{C}$ (and write $\mathrm{slim}_{n\to\infty} f_n(z) = f(z)$) if $f_n(z)$ converges to $f(z)$ uniformly on any compact subset of U.
(b) A sequence of holomorphic functions $f_n : U \longrightarrow \mathbb{C}$ *weakly converges* to a holomorphic function $f : U \longrightarrow \mathbb{C}$ (and write $\mathrm{wlim}_{n\to\infty} f_n(z) = f(z)$) if the Taylor series of $f_n(z)$ at $z = 0$ coefficient-wise converges to the Taylor series of $f(z)$. In other words, for every $k \geq 0$, we have:

$$\lim_{n\to\infty} \frac{d^k}{dz^k}|_{z=0} f_n(z) = \frac{d^k}{dz^k}|_{z=0} f(z).$$

It is easy to see that strong convergence of holomorphic functions implies weak convergence. The converse is not true (see however, Lemma 2.1 below).

The Melvin-Morton-Rozansky (MMR, in short) Conjecture, which was settled by Bar-Natan and the first author in [B-NG], compares the function $f_{K,n}$ of a knot K with the *Alexander polynomial* Δ_K of K, normalized by $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$.

Theorem 1. (The MMR conjecture) [B-NG] *For every knot K we have*

$$\mathrm{wlim}_{n\to\infty}\ f_{K,n}(z) = \frac{1}{\Delta_K(e^z)}.$$

Our sample result is the following analytic form of the MMR Conjecture, which has application in the Generalized Volume Conjecture.

Theorem 2. *For every knot K there exists an open neighborhood U_K of $0 \in \mathbb{C}$ such that in U_K, we have*

$$\mathrm{slim}_{n\to\infty} f_{K,n}(z) = \frac{1}{\Delta_K(e^z)}.$$

Given Theorem 2 one may ask for a full asymptotic expansion of $f_{K,n}(z)$ in terms of powers of $1/n$. In order to formulate our results, let us introduce the notion of strong and weak asymptotic expansions.

Definition 1.2. Fix an open set U of $\mathbb{C}$, and holomorphic functions $f_n : U \longrightarrow \mathbb{C}$ and $R_n : U \longrightarrow \mathbb{C}$.
(a) We will say that the sequence f_n is *strongly asymptotic in U* to the series $\sum_{k=0}^{\infty} R_k(z)\left(\frac{z}{n}\right)^k$, and write

$$f_n(z) \sim^s_{n\to\infty} \sum_{k=0}^{\infty} R_k(z)\left(\frac{z}{n}\right)^k \tag{1}$$

if for every $N \geq 0$ we have:

$$\mathrm{slim}_{n\to\infty}\left(\frac{n}{z}\right)^N\left(f_n(z) - \sum_{k=0}^{N-1} R_k(z)\left(\frac{z}{n}\right)^k\right) = R_N(z). \tag{2}$$

(b) Likewise, we will say that the sequence f_n is *weakly asymptotic in U* to the series $\sum_{k=0}^{\infty} R_k(z)\left(\frac{z}{n}\right)^k$, and write

$$f_n(z) \sim^w_{n\to\infty} \sum_{k=0}^{\infty} R_k(z)\left(\frac{z}{n}\right)^k \tag{3}$$

if for every $N \geq 0$ we have:

$$\mathrm{wlim}_{n\to\infty}\left(\frac{n}{z}\right)^N\left(f_n(z) - \sum_{k=0}^{N-1} R_k(z)\left(\frac{z}{n}\right)^k\right) = R_N(z). \tag{4}$$

Typically, sequences of holomorphic functions $f_n(z)$ do not have asymptotic expansions (or even a limit, as $n \to \infty$). However, sequences that appear in perturbative expansions of *Quantum Field Theory* are generally expected to have asymptotic expansions. In fact asymptotic expansions are generally easier to define (via *Feynman diagram* techniques) than the partition functions $f_{K,n}(z)$ themselves. Even when the partition functions can be defined, the asymptotic expansions is a numerically useful way to approximate them.

In [Ro], Rozansky discovered that the sequence $f_{K,n}(z)$ has a weak asymptotic expansion, where the terms are rational functions in the variable e^z. More precisely, Rozansky proved the following result.

Theorem 3. [Ro] *For every knot K there exists a sequence $P_{K,k}(q) \in \mathbb{Q}[q^{\pm}]$ of Laurent polynomials with $P_{K,0}(q) = 1$ such that*

$$f_{K,n}(z) \sim^w_{n\to\infty} \sum_{k=0}^{\infty} \frac{P_{K,k}(e^z)}{\Delta_K(e^z)^{2k+1}}\left(\frac{z}{n}\right)^k. \tag{5}$$

A different proof, valid for all simple Lie groups, was given in [Ga1], using work of [GK].

Our result is the following stronger version.

Theorem 4. *For every knot K there exists an open neighborhood $\tilde{U}_K$ of $0 \in \mathbb{C}$ such that in $\tilde{U}_K$, we have*

$$f_{K,n}(z) \sim^s_{n\to\infty} \sum_{k=0}^{\infty} \frac{P_{K,k}(e^z)}{\Delta_K(e^z)^{2k+1}} \left(\frac{z}{n}\right)^k. \tag{6}$$

1.2. **The generalized volume conjecture.** In this section we state some new information about the Volume Conjecture; the latter connects two very different approaches to knot theory, namely Topological Quantum Field Theory and Riemannian (mostly Hyperbolic) Geometry.

Conjecture 1. [K, MM] *For every hyperbolic knot K in S^3 we have:*

$$\lim_{n\to\infty} \frac{\log |f_{K,n}(2\pi i)|}{n} = \frac{1}{2\pi} \operatorname{vol}(\rho_{2\pi i}),$$

where $\operatorname{vol}(\rho_{2\pi i})$ *is the* hyperbolic volume *of the the knot complement $S^3 - K$.*

In other words, the sequence $f_{K,n}(2\pi i)$ of complex numbers grows exponentially with respect to n, and the exponential growth-rate is proportional to the volume of a hyperbolic knot.

One can define the volume function $\operatorname{vol}(\rho)$ of every representation $\rho : \pi_1(S^3 \setminus K) \to \mathrm{SL}_2(2, \mathbb{C})$, see [Th], and $\operatorname{vol}(\rho_{2\pi i})$ is exactly the value of this volume function with $\rho_{2\pi i}$ being the discrete faithful representation of the knot group.

The idea of the Generalized Volume Conjecture (formulated in part by Gukov in [Gu]) is that we should use other representations of the knot complement in $\mathrm{SL}(2, \mathbb{C})$. For α nearby $2\pi i$, in a small neighborhood of $\rho_{2\pi i}$ there is a unique (up to conjugation) representation

$$\rho_\alpha : \pi_1(S^3 - K) \longrightarrow \mathrm{SL}(2, \mathbb{C})$$

which satisfies

$$\rho_\alpha(\text{meridian}) = \begin{pmatrix} e^\alpha & \star \\ 0 & e^{-\alpha} \end{pmatrix}. \tag{7}$$

Alas, there is an additional difficulty. Namely, we should distinguish two cases for α near $2\pi i$: either $\alpha/(2\pi i)$ is a rational number *not* equal 1, or otherwise. The Generalized Volume Conjecture for α sufficiently close to $2\pi i$ may now be stated as follows.

Conjecture 2. *If $\alpha/(2\pi i) \in (\mathbb{R} - \mathbb{Q}) \cup 1$ is sufficiently close to $2\pi i$ then*

$$\lim_{n\to\infty} \frac{\log |f_{K,n}(\alpha)|}{n} = c_\alpha \operatorname{vol}(\rho_\alpha), \tag{8}$$

and if $\alpha/(2\pi i) \in \mathbb{Q} - 1$, then

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log |f_{K,n}(\alpha)|}{n} &= c_\alpha \operatorname{vol}(\rho_\alpha), \\ \liminf_{n\to\infty} \frac{\log |f_{K,n}(\alpha)|}{n} &= 0, \end{aligned}$$

where $c_\alpha \neq 0$ are some nonzero constant.

The distinction of $\alpha/(2\pi i)$ being rational or not is a bit with odds with the notion of *hyperbolic Dehn surgery* developed by Thurston in [Th]. When $\alpha/(2\pi i)$ is a rational number, the hyperbolic Dehn surgery theorem associates an orbifold filling to the knot complement whose volume is $\operatorname{vol}(\rho_\alpha)$. Orbifolds are mild generalizations of manifolds. On the other hand, when $\alpha/(2\pi i)$ is irrational, hyperbolic Dehn surgery associates a space which is topologically a 1-point compactification of the knot complement, with volume $\operatorname{vol}(\rho_\alpha)$. In the following, we will refer to the parameter α in the Generalized Volume Conjecture as *the angle*, making contact with standard terminology from hyperbolic geometry.

There are two rather independent parts in the Volume Conjecture:

(a) To show that the limit exists in (8),
(b) To identify the limit with the volume of the corresponding Dehn filling.

At the moment, the Generalized Volume Conjecture is known only for the 4_1 knot and certain values of α; see Murakami, [M1].

One may further ask what happens to the Generalized Volume Conjecture when the angle α is small. For $\alpha = 0$, it is natural to define ρ_0 to be the *trivial* representation. Then for α small enough, there is a unique (up to conjugation) *abelian* $\mathrm{SL}_2(\mathbb{C})$ representation ρ_α that satisfies (7).

Abelian representations have 0 volume (see eg, [CCGLS]). On the other hand, for small enough α, we have $\Delta_K(e^\alpha) \sim \Delta_K(1) = 1$. Thus Theorem 2 implies that

Theorem 5. *For every knot K there exists an open neighborhood U_K of $0 \in \mathbb{C}$, such that for $\alpha \in U_K$, we have:*

$$\lim_{n\to\infty} \frac{\log |f_{K,n}(\alpha)|}{n} = 0 = \mathrm{vol}(\rho_\alpha).$$

In other words, Theorem 2 settles the Generalized Volume Conjecture for small complex angles.

1.3. **The Generalized Volume Conjecture near $2\pi i$.** Our next result states that the volume conjecture can only be barely true.

Theorem 6. *For every knot K and every fixed integer $m \neq 0$*

$$\lim_{n\to\infty} \frac{1}{n} \log |J_{K,n+m}(\exp(2\pi i/n))| = 0.$$

It follows that the *double-scaling limit*

$$\lim_{n,k} \frac{1}{n} \log |J_{K,n}(\exp(2\pi i/k))|$$

when $n, k \to \infty$ and $n/k \to 1$ does not exist, or equals to 0; with the latter case in contradiction with the Volume Conjecture.

Our next result confirms the strange behavior in the Generalized Volume Conjecture when $\alpha/(2\pi i)$ is rational, not equal to 1.

Theorem 7. *For every knot K there exist a neighborhood V_K of $1 \in \mathbb{C}$ such that when $\alpha/(2\pi i) \in V_K$ is rational and not equal to 1, then*

$$\liminf_{n\to\infty} \frac{|f_{K,n}(\alpha)|}{n} = 0.$$

1.4. **Upper bounds for the generalized volume conjecture.**

Theorem 8. *For every knot K with $c+2$ crossings and every $\alpha \in \mathbb{C}$, we have*

$$\limsup_{n\to\infty} \frac{\log |f_{K,n}(\alpha)|}{n} \leq c \log 4 + \frac{c+2}{2} |\Re(\alpha)|.$$

1.5. **Relation with hyperbolic geometry, and optimal bounds.** When $\alpha = 2\pi i$, the upper bound in Theorem 8 is not optimal, and does not reveal any relationship between the lim sup and hyperbolic geometry. Our next theorem fills this gap.

Theorem 9. *For every knot K with $c+2$ crossings we have*

$$\limsup_{n\to\infty} \frac{\log |f_{K,n}(2\pi i)|}{n} \leq \frac{v_8}{2\pi} c,$$

where

$$v_8 = 8\Lambda(\pi/4) \approx 3.66386237670887606 02\ldots$$

is the volume of the regular ideal octahedron*–see* [Th].

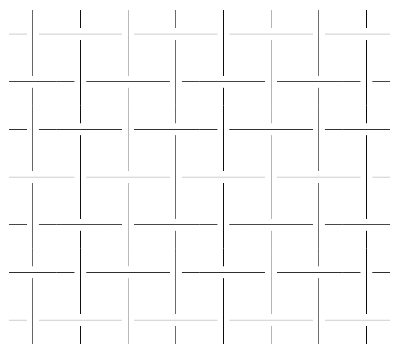
Using an ideal decomposition of a knot complement by placing one octahedron per crossing, it follows that for every knot K with $c+2$ crossings, we have

$$\mathrm{vol}(S^3 - K) \leq v_8 c, \tag{9}$$

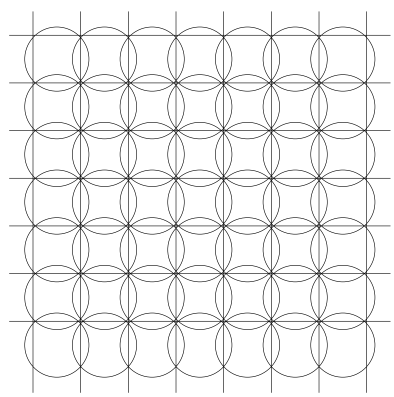
where $\mathrm{vol}(S^3 - K)$ is the hyperbolic volume of the knot complement. On the other hand, if the volume conjecture holds for $\alpha = 2\pi i$, then

$$\lim_{n\to\infty} \frac{\log |f_{K,n}(2\pi i)|}{n} = \frac{1}{2\pi}\mathrm{vol}(S^3 - K) \le \frac{v_8}{2\pi} c.$$

One may ask whether (9) (and therefore, whether the bound in Theorem 9) is optimal. Optimality is at first sight surprising, since it involves all knots (and not just alternating ones) and their number of crossings (which carries little known geometric information). In conversations with I.Agol and D.Thurston, it was communicated to us that the upper bound in (9) is indeed optimal. Moreover a class of knots that achieves (in the limit) the optimal ratio of volume by number of crossings is obtained by taking a large chunk of the following *weave*, and closing it up to a knot:



The complement of the weave has a complete hyperbolic structure associated with the *square tessellation* of the Euclidean plane:



Optimality follows along similar lines as the Appendix of [La], using a stronger estimate for the lower bound of the volume of Haken manifolds, cut along an incompressible surface: If M is a hyperbolic finite volume 3-manifold containing a properly imbedded orientable, boundary incompressible, incompressible surface S, then

$$\mathrm{vol}(M) \ge \mathrm{vol}(\mathrm{Guts}(M - \mathrm{int}(\mathrm{nbd}(S)))),$$

where vol stands for volume, and the Guts terminology are defined in [Ag]. The proof of this stronger statement (of Agol-Dunfield-Storm-W.Thurston [ADST]) uses, among other things, work of Perelman.

The reader may compare (9) with the following result of Agol-Lackenby-D.Thurston [La]: If K is an alternating knot with a planar projection having t twist, then

$$v_3(t-1)/2 < \mathrm{vol}(S^3 - K) < 10 v_3(t-1),$$

where $v_3 = 2\Lambda(\pi/3) \approx 1.01494$ is the volume of the regular ideal tetrahedron. Moreover, the class of knots obtained by Dehn filling on the *chain link* has asymptotic ratio of volume by twist number equal to $10v_3$. The corresponding tessellation of the Euclidean plane is given by the *star of David*.

So far, we have formulated a Generalized Volume Conjecture for α near 0 and α near $2\pi i$, using representations near the trivial one and a discrete faithful, respectively. How can we connect these choices for other complex angles α? A natural answer to this question requires analyzing asymptotics of solutions of difference equations with a parameter. This is a different subject that we will not discuss here; instead we will refer the curious reader to [GG], and forthcoming work of the first author. For a further discussion, see also Section 11.

1.6. **The main ideas and organization of the paper.** In section 2.1 we show that weak convergence plus uniform boundedness implies strong convergence. Thus the strong convergence of Theorems 2 and 4 follows from the weak convergence of Theorems 1 and 3, plus uniform bounds. Uniform bounds for the colored Jones function require large cancellations. In order to control these cancellations, we use the cyclotomic expansion of the colored Jones function of a knot, which is recalled in Section 3. An important point about this expansion is that its kernel can absorb the exponential bounds of the coefficients of the cyclotomic functions; see Sections 4 and 5.

Using a state-sum formula for the colored Jones function, we give in Section 6 bounds for the degrees and coefficients of the n-th colored Jones polynomial. The result is also of independent interest. The important point is that the local weights in the state-sum formula (i.e., the entries of the R-matrix) are Laurent polynomials, given by some ratio of q-factorials. A priori, the bounds of the n-colored Jones function are not good enough to deduce the bounds for the n-th cyclotomic function. However, in Section 7, we use a lemma on the growth-rate of the number of partitions of an integer, in order to deduce the desired bounds for the cyclotomic function. As a corollary, we can deduce the upper bound of Theorem 8.

In the independent Section 8, we give a better bound for the growth-rate of the entries of the R-matrix. The important point is that these entries are ratio of 5 q-factorials, and each q-factorial grows exponentially with rate given by the Lobachevsky function. The q-factorials are arranged in such a way to deduce that the exponential growth-rate of the entries of the R-matrix is given by the volume of an ideal octahedron. Together with our state-sum formulas for the n-th colored Jones polynomial, it results in the upper bound of Theorem 9.

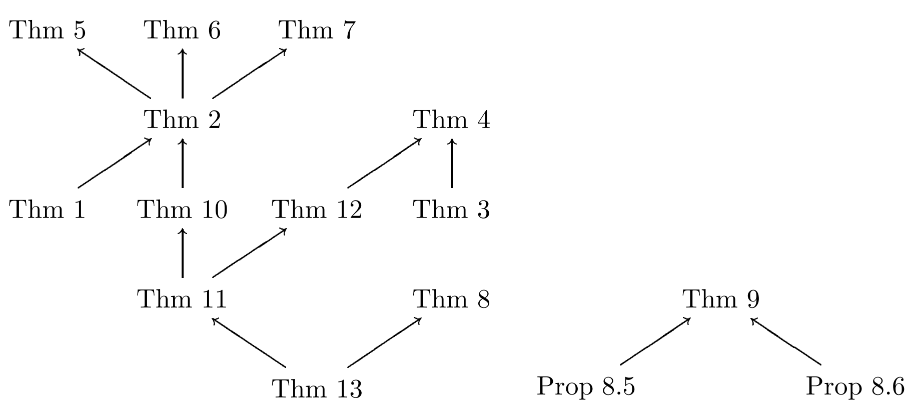
We discuss in Section 9 the proof of Theorems 6 and 7.

In Section 10 we discuss bounds on the degrees and coefficients of q-holonomic functions. Earlier work of the authors implies that the colored Jones and the cyclotomic functions of a knot are q-holonomic.

In Section 11 we discuss some physics ideas related to the various expansions of the colored Jones function.

Finally, in Appendices we establish the Volume Conjecture for the Borromean rings, the Generalized Volume Conjecture for torus knots, and prove a technical statement in the main body of the paper.

The logical dependence of the Theorems is as follows:



1.7. **Acknowledgement.** The authors wish to thank I. Agol, D. Boyd, N. Dunfield, D. Thurston and D. Zeilberger for many enlightening conversations.

2. Weak versus strong convergence

2.1. A lemma from complex analysis.
To prove Theorem 2, we need to improve the weak convergence of Theorem 1 to the strong convergence.

The improvement will use the following lemma on normal families that is sometimes referred to by the name of *Vitali* or *Montel's* theorem. For a reference, see [Hi, Sch]. The lemma exhibits the power of holomorphy, coupled with uniform boundedness.

Lemma 2.1. *If*

$$f_n : \{z \in \mathbb{C} \,:\, |z| < r\} \to \{z \in \mathbb{C} \,:\, |z| \leq M\}$$

is a uniformly bounded sequence of holomorphic functions such that for every $m \geq 0$, we have:

$$\lim_{n\to\infty} f_n^{(m)}(0) = a_m.$$

Then,

- *The limit $f(z) = \lim_n f_n(z)$ exists pointwise for all z with $|z| < r$.*
- *f is holomorphic,*
- *The convergence is uniform on compact subsets, and*
- *For every m, $f^{(m)}(0) = a_m$.*

In other words, weak convergence and uniform boundedness imply strong convergence.

Proof. $\{f_n\}_n$ is uniformly bounded, so it is a normal family, and contains a convergent subsequence $f_j \to f$. Convergence is uniform on compact sets, and f is holomorphic, and for every $m \geq 0$, $\lim_j f_j^{(m)}(0) = f^{(m)}(0) = a_m$.

If $\{f_n\}_n$ is not convergent (uniformly on compact sets), since it is a normal family, then there exist two subsequences that converge to f and g respectively, with $f \neq g$. Applying the above discussion, it follows that f and g are holomorphic functions with equal derivatives of all orders at 0. Thus, $f = g$, giving a contradiction. □

Theorem 2 follows from Lemma 2.1 and the following result, whose proof will be given in Section 4.

Theorem 10. *For every knot K there exists an open neighborhood U_K of $0 \in \mathbb{C}$ and a positive number M such that for $\alpha \in U_K$, and all $n \geq 1$, we have:*

$$|f_{K,n}(\alpha)| < M.$$

2.2. The main difficulty for uniform bounds.
Before we proceed with the proof of Theorem 10, let us point out the main difficulty. As we will see later, $J_{K,n}(q)$ is a Laurent polynomial in q whose span (i.e., powers of its monomials) are $O(n^2)$ and whose coefficients are $e^{O(n)}$. In addition, due to our normalization, $J_{K,n}(1) = 1$. In other words, the $O(n^2)$ many exponentially growing coefficients of $J_{K,n}(q)$ add up to 1. When we evaluate $J_{K,n}(e^{\alpha/n})$, we want to bound the result independent of n. This will happen only if major cancellations occur. How can we control these cancellations? The answer to this is a key cyclotomic expansion of the colored Jones function, which we review next.

3. Two expansions of the colored Jones polynomial

3.1. The loop expansion.
With $q = e^h$, one has

$$J_{K,n}(e^h) = \sum_{i=0}^{\infty} a_{K,i}(n)\, h^i \in \mathbb{Q}[[h]].$$

It turns out that $a_{K,i}(n)$ is a polynomial in n with degree less than or equal to i, see [B-NG]. Hence there are rational numbers $a_{K,i,j}$, depending on the knot K, such that

$$\begin{aligned} J_{K,n}(e^h) &= \sum_{0\leq j\leq i} a_{K,i,j}\, n^j h^i = \sum_{0\leq i, 0\leq j\leq i} a_{K,i,j}(nh)^j h^{i-j} \\ &= \sum_{0\leq j,k} a_{K,j+k,j}(nh)^j h^k. \end{aligned}$$

If we define

$$R_{K,k}(x) = \sum_{0\leq j} a_{K,j+k,j} x^j \in \mathbb{Q}[[x]],$$

then we have the following *loop expansion*

$$J_{K,n}(e^h) = \sum_{k=0}^{\infty} R_{K,k}(nh)h^k \in \mathbb{Q}[[h]] \tag{10}$$

It turns out that $R_{K,k}$'s are rational functions. In fact, the MMR Conjecture states that

$$R_{K,0}(x) = \frac{1}{\Delta_K(e^x)} \in \mathbb{Q}[[x]].$$

More generally, Rozansky [Ro] proves there are Laurent polynomials $P_{K,k}(q) \in \mathbb{Q}[q^{\pm}]$ such that in $\mathbb{Q}[[x]]$,

$$R_{K,k}(x) = \frac{P_{K,k}(e^x)}{\Delta_K(e^x)^{2k+1}}.$$

Remark 3.1. For every i, j, the function $K \to a_{K,i,j}$ is a finite type invariant of degree i. Although the polynomials $P_{K,k}(q)$ are not finite type invariants (with respect to the usual crossing change of knots), they are indeed finite type invariants with respect to a loop move described in [GR]. We will not use these facts in our paper.

3.2. **The cyclotomic expansion.** Habiro [H1] found another interesting expansion, known as the cyclotomic expansion. Although the cyclotomic expansion has a strong geometric flavor, we discuss only its algebraic properties here. Let us define:

$$C_{n,k}(q) = \prod_{j=1}^{k}(q^n + q^{-n} - q^j - q^{-j}), \quad \text{with} \quad C_{n,0}(q) := 1. \tag{11}$$

By successively solving for $H_{K,n}(q)$, beginning with $n = 0$, from the equation

$$J_{K,n}(q) = \sum_{k=0}^{n-1} C_{n,k}(q)\, H_{K,k}(q) \tag{12}$$

we see that the unique solution $H_{K,k}(q) \in \mathbb{Q}(q)$ are *rational functions* in q. A non-trivial result of Habiro [H1] is that for any knot K, the functions $H_{K,k}(q)$ are *Laurent polynomial in q with integer coefficients*: $H_{K,k}(q) \in \mathbb{Z}[q^{\pm 1}]$; see [H1]. We will call the expansion (12) the cyclotomic expansion. Note that $C_{n,k}(q) = 0$ if $k \geq n$, hence the summation in (12) can be assumed from 0 to ∞.

Explicitly, from [H1] one has

$$H_{K,n}(q) = \frac{1}{\{2n+2\}!} \sum_{k=1}^{n+1} (-1)^{n+1-k} \{2k\}\{k\} \begin{bmatrix} 2n+2 \\ n+1-k \end{bmatrix} J_{K,k}(q) \tag{13}$$

where we use the following definition

$$\{n\} := q^{n/2} - q^{-n/2} \quad \text{and} \quad \{n\}! := \prod_{i=1}^{n} \{i\},$$

$$\{a\}_b := \frac{\{a\}!}{\{a-b\}!} = \prod_{j=a-b+1}^{a} \{j\}, \qquad \begin{bmatrix} a \\ b \end{bmatrix} := \frac{\{a\}!}{\{b\}!\{a-b\}!}$$

3.3. **Comparing the cyclotomic and the loop expansion.** In the loop expansion, as well as in the cyclotomic expansion, one should treat q^n and q (where n is the color) as two independent variables. Consider two independent variables z (standing for α) and y (standing for α/n). Let us define the following biholomorphic functions

$$\begin{aligned} c_k(z,y) &= \prod_{j=1}^{k} (e^z + e^{-z} - e^{jy} - e^{-jy}), \\ h_{K,k}(z,y) &= c_k(z,y) H_{K,k}(e^y). \end{aligned}$$

The cyclotomic expansion says that for every n we have:

$$f_{K,n}(\alpha) = \sum_{k=0}^{\infty} h_{K,k}(\alpha, \alpha/n) \in \mathbb{Q}[[\alpha]]. \tag{14}$$

The loop expansion is a Taylor expansion in α/n, so we will consider Taylor expansion in y (around 0):

$$h_{K,k}(z,y) = \sum_{p=0}^{\infty} d_{k,p}(z)\, y^p,$$

where $d_{k,p}(z)$ is holomorphic for $z \in \mathbb{C}$.

Comparing the loop and the cyclotomic expansion (Equations (5) and (14)), we obtain that:

Lemma 3.2. *For every knot K and every $p \in \mathbb{N}$ we have*

$$R_{K,p}(x) = \sum_{k=0}^{\infty} d_{k,p}(x) \in \mathbb{Q}[[x]]. \tag{15}$$

as formal power series in x.

4. A reduction of Theorem 10 to estimates of the cyclotomic function

4.1. **Uniform bounds of the colored Jones function.** In this section we will deduce Theorem 10 from estimates of the degree and the coefficients of the cyclotomic expansion of the knot. These estimates will be established in a later chapter.

By definition $f_{K,n}(\alpha) = J_{K,n}(e^{\alpha/n})$, hence equation (12) gives that

$$f_{K,n}(\alpha) = \sum_{k=0}^{n-1} C_{n,k}(e^{\alpha/n}) H_{K,k}(e^{\alpha/n}). \tag{16}$$

To have upper bounds for $|f_{K,n}(\alpha)|$ we will need bounds for $H_{K,n}(e^{\alpha/n})$ and the "kernel" $C_{n,k}(e^{\alpha/n})$ (the kernel does not depend on the knot K).

Definition 4.1. For a Laurent polynomial $f(q) = \sum_k a_k q^k$, we define its l^1-norm by

$$||f||_1 = \sum_k |a_k|.$$

The proof of the following Theorem, which gives bounds for the degrees and the l^1-norm of $H_{K,n}$, will be given in Chapter 7.

Theorem 11. *For every knot K, there are positive constants A_0, A_1 (depending on K) such that*

(a)
$$H_{K,n}(q) = \sum_{j=-A_0 n^2}^{A_0 n^2} b_{j,n} q^j.$$

and

(b)
$$||H_{K,n}||_1 \leq A_1^n.$$

The next lemma follows from an elementary estimate.

Lemma 4.2. *Suppose $|\alpha| \leq 1$.*

(a) *For every knot K there is a constant A_2 such that for every $0 \leq k \leq n$, we have*

$$|H_{K,k}(e^{\alpha/n})| \leq (A_2)^k.$$

(b) *There is a constant $A_3 > 0$ such that every $0 \leq k \leq n$ we have:*

$$|C_{n,k}(e^{\alpha/n})| \leq (A_3)^k |\alpha|^k.$$

Proof. (a) By Theorem 11(a),

$$H_{K,k}(e^{\alpha/n}) = \sum_{j=-A_0 k^2}^{A_0 k^2} b_{j,k} e^{j\alpha/n}.$$

From the bounds for j and $k \leq n$ one has that $j/n \leq A_0 k$, hence $|e^{j\alpha/n}| \leq \exp(A_0 k \Re(\alpha)) \leq \exp(k A_0)$. From the above equation one has

$$|H_{K,k}(e^{\alpha/n})| \leq ||H_{K,k}||_1 \, (\exp A_0)^k.$$

Using Theorem 11, it is enough to take $A_2 = A_1 \exp(A_0)$.

(b) By definition,

$$C_{n,k}(e^{\alpha/n}) = \prod_{j=1}^{k} (e^\alpha + e^{-\alpha} - e^{j\alpha/n} - e^{-j\alpha/n}) = \prod_{j=1}^{k} (g(\alpha) - g(j\alpha/n)),$$

where $g(z) = e^z + e^{-z}$. One has $g'(z) = e^z - e^{-z}$, hence for z on the interval connecting α and $j\alpha/n$, with $0 \leq j \leq n$, one has $|g'(z)| \leq 2\exp(|\alpha|) < 2e$. By the mean value theorem, we have, for $0 \leq j \leq k \leq n$,

$$|g(\alpha) - g(j\alpha/n)| \leq 2e|\alpha - j\alpha/n| \leq 2e|\alpha|.$$

It follows that

$$|C_{n,k}(e^{\alpha/n})| \leq (2e)^k |\alpha|^k.$$

It is enough to take $A_3 = 2e$. □

4.2. **Proof of Theorem 10, assuming Theorem 11.** It follows from Lemma 4.2 that for $0 \leq k \leq n$ and $|\alpha| \leq 1$, we have:

$$|C_{n,k}(e^{\alpha/n}) H_{K,k}(e^{\alpha/n})| \quad \leq \quad |\alpha A_2 A_3|^k.$$

Let us choose U_K to be the disk centered at the 0, with radius $1/(2A_2A_3 + 1)$, then $|\alpha A_2 A_3| < 1/2$ for $\alpha \in U_K$. Equation (16) and the above estimate imply that for all n and all $\alpha \in U_K$, we have:

$$|f_{K,n}(\alpha)| \leq \sum_{k=0}^{n-1} |C_{n,k}(e^{\alpha/n}) H_{K,k}(e^{\alpha/n})| \leq \sum_{k=0}^{n-1} (1/2)^k < 2.$$

which concludes the proof of Theorem 10, assuming Theorem 11. □

5. A reduction of Theorem 4 to estimates of the cyclotomic function

5.1. **Some estimates.** The following is a higher order version of Lemma 4.2. The proof is similar.

Lemma 5.1. *Suppose* $|\alpha| \le 1$.

(a) *For* $1 \le k \le n$, $0 \le l$, *and* y *on the interval from 0 to* α/n *we have*

$$\left| \frac{\partial^l}{\partial y^l} c_k(\alpha, y) \right| < (A_3)^k k^{3l} |\alpha|^{k-l}.$$

(b) *For any* $y \in \mathbb{C}, |y| < 1/n$ *and* $1 \le k \le n$ *we have*

$$\left| \frac{\partial^l}{\partial y^l} H_{K,k}(e^y) \right| < A_2^k (A_0 k^2)^l.$$

Proof. (a) We have $c_k(\alpha, y) = \prod_{j=1}^k g_j$, where

$$g_j(y) = e^\alpha + e^{-\alpha} - e^{jy} - e^{-jy}.$$

Here we suppress the variable α. The l-th derivative (with respect to y) of c_k is the sum of k^l term, each of the form

$$t = \prod_{j=1}^k g_j^{(l_j)}, \quad \text{with} \quad \sum_{j=1}^k l_j = l. \tag{17}$$

We will estimate each term t. There are at least $k-l$ indices j from $\{1, \dots, k\}$ such that $l_j = 0$. Choose a set $I \subset \{1, \dots, k\}$ such that $l_j = 0$ for $j \in I$ and I has cardinality $k - l$. Note that if $l > k$, then I is empty. We consider two cases, $j \in I$ and $j \notin I$.

Suppose $j \in I$. Then $g_j^{(l_j)} = g_j = e^\alpha + e^{-\alpha} - e^{jy} - e^{-jy}$. Since $j \le k \le n$, the interval connecting α and jy lies totally in the disk of radius $|\alpha|$ (remember that $|y| \le |\alpha|/n$). As in the proof of Lemma 4.2, we have

$$|(e^\alpha + e^{-\alpha}) - (e^{jy} + e^{-jy})| \le (2e)|\alpha|.$$

Taking the product we have

$$\left| \prod_{j \in I} g_j^{(l_j)} \right| \le (2e)^{k-l} |\alpha|^{k-l}. \tag{18}$$

Note that this also holds true when I is empty, as long as $2e|\alpha| \ge 1$.

Now suppose $j \notin I$. We have

$$g_j^{(l_j)} = e^\alpha + e^{-\alpha} - e^{jy} j^{l_j} - e^{-jy}(-j)^{l_j}$$

It is clear $|e^{\pm\alpha}| < e$. Since $|j| \le |k|$ and $|jy| < |\alpha|$, we have $|e^{\pm jy}(\pm j)^{l_j}| < e k^{l_j}$. Hence

$$\left| g_j^{(l_j)} \right| < 2e(1 + k^{l_j}) < 2e k^{2l_j}.$$

Taking the product, we have

$$\left| \prod_{j\{1,\dots,k\}\setminus I} g_j^{(l_j)} \right| < (2e)^l k^{2l}. \tag{19}$$

Combining (18) and (19), and recalling that we have in total k^l terms of the form (17), we get the result with $A_3 = 2e$.

(b) By Theorem 11(a),

$$\frac{\partial^l}{\partial y^l} H_{K,k}(e^y) = \sum_{j=-A_0k^2}^{A_0k^2} b_{j,k} e^{jy} j^l.$$

From the bounds for j and $k \leq n$ one has $|e^{jy}| \leq \exp(A_0 k)$ and $j^l \leq (A_0 k^2)^l$. From the above equation one has

$$|\frac{\partial^l}{\partial y^l} H_{K,k}(e^y)| \leq ||H_{K,k}||_1 \, (\exp A_0)^k (A_0 k^2)^l.$$

Using Theorem 11, it is enough to take $A_2 = A_1 \exp(A_0)$. □

Corollary 5.2. *For every knot K there are positive constants A_4, A_5 such that*

(a) *for $0 \leq k$, $0 \leq N$, and $|\alpha| < 1$ and y is on the interval from 0 to α/k, we have*

$$\left| \frac{\partial^N}{\partial y^N} h_{K,k}(\alpha, y) \right| < |\alpha A_4|^{k-N} (A_5 k)^N.$$

(b) *for $0 \leq k$, $0 \leq N$, and $|\alpha| < 1$ and every positive integer n, we have*

$$\left| h_{K,k}(\alpha, \alpha/n) - \sum_{p=0}^{N-1} d_{k,p}(\alpha) \, (\alpha/n)^p \right| < \frac{1}{N!} \left(\frac{\alpha}{n}\right)^N \; |\alpha A_4|^{k-N} (A_5 k)^N.$$

Proof. (a) The N-th derivative of $h_{K,k}(\alpha, y)$, which is the product of $c_k(\alpha, y)$ and $H_{K,k}(e^y)$, is the sum of 2^N terms, each is of the form

$$\frac{\partial^l}{\partial y^l} c_k(\alpha, y) \, \frac{\partial^{N-l}}{\partial y^{N-l}} H_{K,k}(e^y).$$

Using the Lemma 5.1, the absolute value of the above term is bounded by $|\alpha|^{k-l} (A_2 A_3)^k (A_0)^{N-l} k^l$, which, in turn, is less than $|\alpha|^{k-N} (A_2 A_3)^k (A_0)^N k^N$. Hence, multiplied by 2^N we get

$$\left| \frac{\partial^l}{\partial y^l} c_k(\alpha, y) \right| < 2^N \times |\alpha|^{k-N} (A_2 A_3)^k (A_0)^N k^N = (\alpha A_2 A_3)^{k-N} (2 A_0 A_2 A_3 k)^N.$$

It is enough to take $A_4 = A_2 A_3$ and $A_5 = 2 A_0 A_2 A_3$.

(b) By Taylor's Theorem,

$$\left| h_{K,k}(\alpha, \alpha/n) - \sum_{p=0}^{N-1} d_{k,p}(\alpha) \, (\alpha/n)^p \right| < \frac{1}{N!} \left(\frac{\alpha}{n}\right)^N \max \left| \frac{\partial^N}{\partial y^N} h_{K,k}(\alpha, y) \right|,$$

where max is taken when y is on the interval connecting 0 and α/n. Using the estimate of part (a), we get the result. □

5.2. **Proof of Theorem 4 assuming Theorem 11.** To simplify notation, let us define, for a knot K,

(20) $$f^{[N]}_{K,n}(z) := J_{K,n}(e^{z/n}) - \sum_{k=0}^{N-1} \frac{P_{K,k}(e^z)}{\Delta_K(e^z)^{2k+1}} \left(\frac{z}{n}\right)^k.$$

Theorem 4 follows from Theorem 3, Lemma 2.1 and the following uniform bound.

Theorem 12. *For every knot K there exists an open neighborhood $\tilde{U}_K$ of $0 \in \mathbb{C}$ such that for every $N \geq 0$ there exists a positive number M_N such that for $\alpha \in U_K$, and all $n \geq 0$, we have:*

$$\left| \left(\frac{n}{\alpha}\right)^N f^{(N)}_{K,n}(\alpha) \right| < M_N.$$

Proof. (of Theorem 12, assuming Theorem 11)

We have the following identities, where the second follows from (14) and (15):

$$
\begin{aligned}
f^{[N]}_{K,n}(\alpha) &= f_{K,n}(\alpha) - \sum_{p=0}^{N-1} R_{K,p}(\alpha)\,\left(\frac{\alpha}{n}\right)^p \\
&= \sum_{k=0}^{\infty} h_{K,k}(\alpha, \frac{\alpha}{n}) - \sum_{p=0}^{N-1}\left(\sum_{k=0}^{\infty} d_{k,p}(\alpha)\right)\left(\frac{\alpha}{n}\right)^p \\
&= \sum_{k=0}^{\infty}\left[h_{K,k}(\alpha, \frac{\alpha}{n}) - \sum_{p=0}^{N-1} d_{k,p}(\alpha)\left(\frac{\alpha}{n}\right)^p\right]
\end{aligned}
$$

Using the estimate in Corollary 5.2, we see that

$$
\left|(\frac{n}{\alpha})^N f^{[N]}_{K,n}(\alpha)\right| < \frac{1}{N!}\sum_{k=0}^{\infty}(\alpha A_4)^{k-N}(A_5 k)^N.
$$

If $|\alpha A_4| < 1$, the series of the right hand side is absolutely convergent. It is enough to take $\tilde{U}_K$ to be the disk centered at 0 with radius $1/(2A_4+1)$. This proves Proposition 12, assuming Theorem 11. □

6. Bounds for the degree and coefficients of the colored Jones function

6.1. **Bounds for the degree.** For a Laurent polynomial $f(q) = \sum_{k=m}^{M} a_k q^k$, with $a_m a_M \neq 0$, let us define $\deg_+(f) = M$ and $\deg_-(f) = m$. The bounds of degrees of q in the colored Jones polynomial was studied by the second author in [Le2]. In [Le2], the framed version of the colored Jones was used, the variable $t = q^{1/4}$ was used, and the normalization of $J_{K,n}$ was chosen so that it is equal to $[n]$ times our $J_{K,n}$. Suppose the knot K has a planar projection with $c+2$ crossings. Let ω be the writhe number, i.e. the number of positive crossing minus the number of negative ones. Let $s_\pm$ denote the number of $\pm$-oriented circles, as in [Le2, Prop.2.1]. Then by Proposition 2.1 of [Le2], taking into account the change of variable, the framing, and the normalization, one has the following bounds for the degrees of $J_{K,n}(q)$.

Proposition 6.1. *With the above notations, one has*

$$
\begin{aligned}
\deg_+(J_{K,n}) &\leq \frac{(c+2)(n-1)^2 + 2(n-1)(s_+-1) - \omega(n^2-1)}{4} \\
\deg_-(J_{K,n}) &\geq -\frac{(c+2)(n-1)^2 + 2(n-1)(s_--1) + \omega(n^2-1)}{4}.
\end{aligned}
$$

6.2. **Bounds for the coefficients.** For Laurent polynomials $f \in \mathbb{Z}[q^{\pm 1/4}]$, we use the same definition of l^1-norm: it is the sum of the absolute values of the coefficients. Observe that

$$
||f+g||_1 \leq ||f||_1 + ||g||_1, \qquad ||fg||_1 \leq ||f||_1||g||_1. \tag{21}
$$

Since $||\{j\}||_1 = 2$, we have, for $k \leq n$

$$
||\{a\}_k||_1 = ||\prod_{j=a-k+1}^{a}\{j\}||_1 \leq 2^k \leq 2^n \tag{22}
$$

It is known that the quantum binomial $\left[\begin{array}{c} m \\ k \end{array}\right]$ is a Laurent polynomial in $q^{1/2}$ with *positive* integer coefficients, hence its l^1-norm is obtained by putting $q^{1/2} = 1$, which is the classical quantum binomial $\binom{m}{k}$. One has, if $m \leq n$,

$$
||\left[\begin{array}{c} m \\ k \end{array}\right]||_1 = \binom{m}{k} \leq 2^m \leq 2^n \tag{23}
$$

Theorem 13. *For every knot K of $c+2$ crossings and every n we have:*

$$||J_{K,n}||_1 \leq n^c 4^{cn}. \tag{24}$$

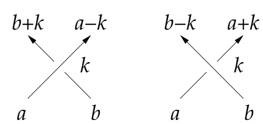
Proof. The proof of the Theorem is pretty simple using the state sum definition of the colored Jones polynomial: The colored Jones polynomial is the sum, over all states, of the weights of the states. There are c^n states, the weight of each is the product of several q-factorials and q-binomial coefficients for which an upper can be easily found.

Let us go to the details.

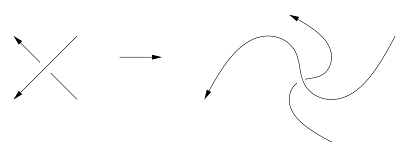
The knot K is the closure of a $(1,1)$-tangle T (or long knot), with orientation given by the direction from the bottom boundary point to the top boundary point. The crossing points of the diagram of T (on the standard 2-plane) break T into $2c+5$ arcs, two of them are *boundary* (i.e. each contains a boundary point of T). The two crossings adjacent to the boundary arcs are called *boundary crossings.*

To get from $c+2$ to c in the estimate, we will choose the $(1,1)$-tangle T such that (1) when going along T, starting at the bottom boundary point, we must pass the very first crossing (resp. very last crossing) by an overpass (respectively, an underpass) and (2) the two strands at each crossing are pointing upwards, as in the following figure:

(25)



Here is how to get such a $(1,1)$-tangle T. Consider a diagram of K on a 2-sphere S^2. The $c+2$ crossings break the knot diagram into $2c+4$ arcs. At each crossing we have an overpass and an underpass. When we go along the knot starting at some point, following the direction of the orientation, we pass through all these underpasses and overpasses. Hence there must be an arc which starts at an underpass and ends at an overpass, assuming there is at leat one crossing. Remove from S^2 a small disk which is a small neighborhood of a point inside this arc. What is left is a long knot diagram on a disk, which can also be considered as a $(1,1)$-tangle diagram in the strip $\mathbb{R}\times[0,1]$ in the standard 2-plane which satisfies requirement (1). Using the isotopy of the form



which moves crossings (positive or negative) into standard upright position, we get the desired $(1,1)$-tangle.

A state $\mathbf{k}$ is an assignment of numbers, called the colors, to the crossings of the diagram of T, where each color is in $\{0,\dots,n-1\}$. For a fixed state we will color the $2c+5$ arcs as follow. First color the bottom boundary arc by 0. Going along the diagram of T from the bottom boundary point, if we are on an arc of color a and pass a crossing, the next arc will have color $a+k$ or $a-k$, according as the pass is an underpass or an overpass, see Figure 25. Here k is the color of the crossing.

We will only consider states such that the colors of arcs are between 0 and $n-1$ and the color of the top boundary arc is 0. The under/overpass configuration at the two boundary crossings ensures that the two boundary crossings have color 0, otherwise the arcs next to the two boundary arcs would have negative colors. It follows that the number of states is at most n^c.

The weights of the positive crossing (on the left) and negative crossing (on the right in Figure 25) are

$$R_+(n;a,b,k) = (\text{unit}) \begin{bmatrix} b+k \\ k \end{bmatrix} \{n-1+k-a\}_k, \tag{26}$$

$$R_-(n;a,b,k) = (\text{unit}) \begin{bmatrix} a+k \\ k \end{bmatrix} \{n-1+k-b\}_k, \tag{27}$$

where (unit) stands for $\pm$ a power of $q^{\pm 1/4}$, which does not affect the l^1 norm. Note that both $a+k$ and $b+k$ in the above formulas are between 0 and $n-1$.

The weight of a maximum/minimum point is a also $\pm$ a power of $q^{\pm 1/4}$, whose exact formula is not important for us. Let $F(n,\mathbf{k})$ denote the product of weights of all the crossings and all the extreme points. Then

$$J_{K,n}(q) = \sum_{\mathbf{k}} F(n,\mathbf{k}). \tag{28}$$

Using the estimates (23) and (22), we see that $||R_\pm(n;a,b,k)||_1 \leq 4^n$. Since the weight of the two boundary crossing is just a unit, the l^1 norm of $F(n,\mathbf{k})$ is less than 4^{cn}. From (28) and the fact that there are n^c states, we get $||J_{K,n}||_1 \leq n^c 4^{cn}$. □

Since there is a constant b such that $n^c \leq b^n$, we have the following.

Theorem 14. *For every knot K, there is a constant A_6 such that for every positive integer n,*

$$||J_{K,n}||_1 \leq (A_6)^n.$$

As an application of our bounds, we can prove Theorem 8.

Proof. (of Theorem 8) Fix a knot with $c+2$ crossings. The bounds for the degrees of $J_{K,n}$(see Proposition 6.1), allow us to write

$$J_{K,n}(q) = \sum_j a_{n,j} q^j$$

where $|j| \leq n^2(c+2+|w|)/4 + O(n)$. For such j, we have:

$$|e^{j\alpha/n}| = e^{\Re(j\alpha)/n} \leq e^{(c+2+|w|)n/4+O(1))|\Re(\alpha)|}.$$

Using Theorem 13 we get

$$|J_{K,n}(e^{\alpha/n})| \leq n^c 4^{cn} e^{(c+2+|w|)n/4+O(1))|\Re(\alpha)|}.$$

Thus,

$$\frac{1}{n}\log|f_{K,n}(\alpha)| \leq c\log 4 + \frac{c+2+|w|}{4}|\Re(\alpha)|) + O\left(\frac{\log n}{n}\right).$$

The result follows from the observation that $|\omega| \leq c+2$, since $c+2$ is the total number of crossings. □

7. Bounds for the degree and coefficients of the cyclotomic function

The goal of this Section is to prove Theorem 11.

7.1. The bound for degrees of $H_{K,n}$.

Note that

$$\deg_\pm(fg) = \deg_\pm(f) + \deg_\pm(g), \quad \text{and} \quad \deg_+(f+g) \leq \max(\deg_+(f), \deg_+(g)).$$

From $\deg_\pm\{k\} = \pm k/2$, we get

$$\deg_\pm(\{k\}!) = \pm k(k+1)/4, \qquad \deg_\pm(\begin{bmatrix} n \\ k \end{bmatrix}) = \pm k(n-k)/2.$$

From these and Equation (13) we get

$$\deg_+(H_{K,n}(q)) \leq \max_{1\leq k\leq n+1}\left(-\frac{(2n+2)(2n+3)}{4} + k + \frac{k}{2} + \frac{(n+1+k)(n+1-k)}{2} + \deg_+(J_{K,k})\right).$$

Using Proposition 6.1 for the upper bound of $\deg_+(J_{K,k})$, after a simplification, we get

$$\deg_+(H_{K,n}(q)) \leq \max_{1\leq k\leq n+1}\left(-\frac{n(n+3)}{2} + \frac{c(k-1)^2}{4} + \frac{(k-1)s_+}{2} + \frac{|\omega|(k^2-1)}{4}\right).$$

The right hand side reaches maximum when $k=n+1$. Using $\omega \leq c+2$, we have

$$\deg_+(H_{K,n}(q)) \leq n^2 c/2 + n(s_+ + c - 1)/2.$$

A similar calculation shows that

$$\deg_-(H_{K,n}(q)) \geq -(n^2c/2 + n(s_- + c - 1)/2).$$

If we choose A_0 bigger than c and $|s_\pm + c - 1|$, then we have $|\deg_\pm(H_{K,n})| \leq A_0 n^2$. This proves the first statement of Theorem 11.

7.2. **The bound for the l^1-norm of $H_{K,n}$.** Multiply both sides of (13) by $\{2n+2\}!$, then use (23) and Theorem 14, we see that there is a constant A_7 such that

$$||\{2n+2\}! H_{K,n}(q)||_1 \leq (A_7)^n. \tag{29}$$

The polynomials

$$\tilde{H}_{K,n}(q) := q^{A_0 n^2} H_{K,n}(q) \qquad \text{and} \qquad g(q) := \tilde{H}_{K,n}(q) \prod_{j=1}^{2n+2} (1-q^j)$$

have only non-negative degrees in q, with $\deg_+(\tilde{H}_{K,n}) \leq 2A_0 n^2$:

$$\tilde{H}_{K,n}(q) = \sum_{k=0}^{2A_0 n^2} a_k q^k \tag{30}$$

Since $g(q)$ is the product of the polynomial on the left hand side of (29) and a power of q, we have

$$||g(q)||_1 \leq (A_7)^n. \tag{31}$$

A priori, this estimate is weak and implies an exponential upper bound on the Mahler measure of $H_{K,n}(q)$, and a doubly exponential upper bound on the l^1-norm of $H_{K,n}(q)$. The following argument was communicated to us by D. Boyd: Since

$$\tilde{H}_{K,n}(q) = g(q) \frac{1}{\prod_{k=j}^{2n+2}(1-q^j)},$$

we have that

$$a_k = \sum_{i=0}^{k} b_i c_{k-i}, \quad \text{where} \quad g(q) = \sum_k b_k q^k \quad \text{and} \quad \frac{1}{\prod_{k=1}^{2n+2}(1-q^k)} = \sum_{k=0}^{\infty} c_k q^k.$$

Note that c_k is the number of partitions of k of length $\leq 2n+2$. Hence $0 \leq c_{k-1} \leq c_k$, and $c_k \leq p_k$, where p_k is the number of partitions of k. Using the growth rate of p_k (see [An]), we see that there is a constant A_8 such that

$$p_k < (A_8)^{\sqrt{k}}. \tag{32}$$

The upshot is that the exponent is $\sqrt{k}$. Now we can easily obtain the desired upper bounds for $||\tilde{H}_{K,n}||_1$:

Since $a_k = \sum_{i=0}^k b_i c_{k-i}$ we have

$$\begin{aligned} |a_k| &\leq \sum_{i=0}^{k} |b_i| c_{k-i} \leq \left(\sum_{i=0}^{k} |b_i| \right) c_k \leq ||g(q)||_1 c_k \\ &\leq (A_7)^n (A_8)^{n\sqrt{2A_0}} \qquad \text{by (31), (32) and } k \leq 2A_0 n^2 \end{aligned}$$

It follows that, for $n \geq 1$,

$$||\tilde{H}_{K,n}||_1 \leq \sum_{k=0}^{2A_0 n^2} |a_k| \leq 2A_0 n^2 (A_7)^n (A_8)^{n\sqrt{2A_0}} \leq (A_1)^n,$$

for appropriate A_1. This completes the proof of Theorem 11.

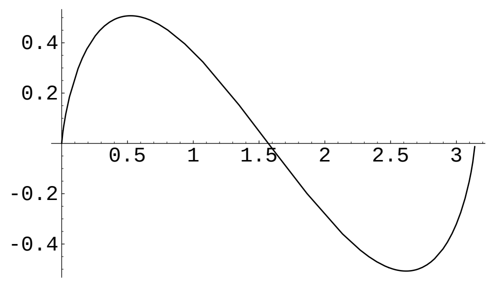
8. Growth rates of R-matrices and the Lobachevsky function

8.1. **The Lobachevsky function.** In Section 6 we got a simple but crude estimate for the l^1-norm of the R-matrices, which are a ratio of five quantum factorials. In this largely independent section we will give refined (and optimal) estimates for the growth rate of the R-matrices. These estimates reveal the close relationship between hyperbolic geometry and the asymptotics of the quantum factorials.

Recall that the *Lobachevsky function* is given by

$$\Lambda(z) = -\int_0^z \log|2\sin x|dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2nz)}{n^2}$$

The Lobachevsky function is an odd, periodic function with period π. Its graph for $z \in [0, \pi]$ is given by:



Definition 8.1. If $f(q) \in \mathbb{Z}[q^{\pm 1/4}]$, let us denote by $\mathrm{ev}_n(f)$ the *evaluation* of f at $q^{1/4} = e^{\pi i/(2n)}$.

For $0 \leq k \leq n$ we have

$$\mathrm{ev}_n|\{k\}| = |e^{k\pi i/n} - e^{-k\pi i/n}| = 2\sin(k\pi/n),$$

hence,

$$\log(\mathrm{ev}_n|(\{j\}!)|) = \sum_{k=1}^{j}\log|2\sin(k\pi/n)|,$$

which is very closely related to a Riemann sum of the integral in the definition of the Lobachevsky function. It is not surprising to have the following.

Proposition 8.2. *For every $\alpha \in (0, 1)$ we have:*

$$\log|\mathrm{ev}_n(\{\lfloor \alpha n\rfloor\}!)| = -\frac{n}{\pi}\Lambda(\pi\alpha) + O(\log n).$$

Here $O(\log n)$ is a term which is bounded by $C\log n$ for some constant C independent *of α.*

Remark 8.3. The proof reveals an asymptotic expansion of the form:

$$\mathrm{ev}_n(\{\lfloor \alpha n\rfloor\}!) \sim n^{\theta}\exp\left(-\frac{n}{\pi}\Lambda(\pi\alpha)\right)\left(C_0 + \frac{C_1}{n} + \frac{C_2}{n^2} + \dots\right)$$

for explicitly computable constants C_i.

Proof. Recall the *Euler-MacLaurin summation formula*, with error term (see for example, [O, Chpt .8]):

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{1}{2}f(a) + \frac{1}{2}f(b) + \sum_{k=1}^{m-1}\frac{B_{2k}}{(2k)!}(f^{(2k-1)}(b) - f^{(2k-1)}(a)) + R_m(a, b, f)$$

where

$$|R_m(a, b, f)| \leq (2 - 2^{1-2m})\frac{|B_{2m}|}{(2m)!}\int_a^b |f^{(2m)}(x)|dx,$$

and B_k is the kth *Bernoulli number* given by the generating series:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k\frac{x^k}{k!}.$$

Applying the above formula to for $m = 1$ to $f(x) = \log(2\sin x\pi/n)$, we have:

$$\begin{aligned}\log(\prod_{k=1}^{\lfloor \alpha n\rfloor} 2\sin(k\pi/n)) &= \frac{1}{2}(f(1)+f(\lfloor \alpha n\rfloor)) + \int_1^{\lfloor \alpha n\rfloor} \log(2\sin(t\pi/n))dt + R_1(1,\lfloor \alpha n\rfloor, f)\\ &= \frac{1}{2}(f(1)+f(\alpha n) + \int_1^{\alpha n} \log(2\sin(t\pi/n))dt + R_1(1,\lfloor \alpha n\rfloor, f) + \epsilon(\alpha,n)\\ &= \frac{1}{2}(f(1)+f(\alpha n)) + \frac{n}{\pi}\int_{\pi/n}^{\pi\alpha} \log|2\sin(u)|u + R_1(1,\lfloor \alpha n\rfloor, f) + \epsilon(\alpha,n)\\ &= \frac{1}{2}(f(1)+f(\alpha n)) + \frac{n}{\pi}\left(-\Lambda(\pi\alpha)+\Lambda(\frac{\pi}{n})\right) + R_1(1,\lfloor \alpha n\rfloor, f) + \epsilon(\alpha,n).\end{aligned}$$

Now,

$$\frac{1}{2}|f(1)+f(\alpha n)| = O(\log n), \qquad \text{and} \qquad |\epsilon(\alpha,n)| = O(1).$$

Moreover, $f'(x) = -\frac{\pi}{n}\cot(\pi x/n)$ and $f''(x) = \frac{\pi^2}{n^2}(\csc(\pi x/n))^2$, and $(\csc x)^2 = 1/x^2 + 1/3 + x^2/15 + O(x^3)$. Thus $\int_{\pi/n}^{\lfloor \alpha n\rfloor} |\csc x|dx = O(n^2)$, and

$$|R_1(1,\alpha n, f)| = O(1).$$

Furthermore, using the asymptotic expansion of $\Lambda(z)$ for $z \in (0,\pi)$:

$$\Lambda(z) = z - z\log(2z) + \sum_{k=1}^{\infty} \frac{B_k}{2k}\frac{(2z)^{2k+1}}{(2k+1)!},$$

it follows that

$$\frac{n}{\pi}|\Lambda(\frac{\pi}{n})| = O(\log n).$$

The result follows. □

Corollary 8.4. *For every $\alpha \in (0,1)$ and any fixed number d we have:*

$$\log|\mathrm{ev}_n(\{\lfloor \alpha n + d\rfloor\}!)| = -\frac{n}{\pi}\Lambda(\pi\alpha) + O(\log n).$$

Proof. There is $\varepsilon > 0$ such that for big enough n, we have $\varepsilon \le x/n \le 1-\varepsilon$ for every integer x between $\lfloor \alpha n\rfloor$ and $\lfloor \alpha n + d\rfloor$. For such x, we have $0 < 2\sin\varepsilon\pi < 2\sin x\pi < 2$, and hence there is a constant M such that $|\log 2\sin x\pi| < M$. There are at most $|d|+1$ such values of x. Hence the difference between $\log|\mathrm{ev}_n(\{\lfloor \alpha n + d\rfloor\}!)|$ and $\log|\mathrm{ev}_n(\{\lfloor \alpha n\rfloor\}!)|$ by absolute value is less than $(|d|+1)M$, a constant. The result follows. □

8.2. **Asymptotics of the R-matrix using ideal octahedra.** Since the entries of the R-matrix are given by ratios of five quantum factorials (see Equations (26)), Proposition 8.2 gives a formula for the asymptotic behavior of the entries of the R-matrix when evaluated at $e^{2\pi i/n}$. This is the content of the next proposition.

Proposition 8.5. (a) Suppose that α, β, κ are real numbers that satisfy the inequalities

$$\alpha,\beta,\kappa \in [0,1] \qquad 0 \le \beta+\kappa \le 1, \qquad 0 \le \alpha-\kappa \le 1. \tag{33}$$

Then the following limit exists

$$r_+(\alpha,\beta,\kappa) := \lim_{n\to\infty}\frac{1}{n}|\mathrm{ev}_n(R_+(n;\lfloor n\alpha\rfloor,\lfloor n\beta\rfloor,\lfloor n\kappa\rfloor))|, \tag{34}$$

and is equal to

$$r_+(\alpha,\beta,\kappa) = [-\Lambda(\pi(\beta+\kappa)) + \Lambda(\pi\beta) + \Lambda(\pi\kappa) - \Lambda(\pi\alpha) + \Lambda(\pi(\alpha-\kappa))]/\pi. \tag{35}$$

(b) $r_+(\alpha,\beta,\kappa)$ equals to $1/\pi$ times the volume of an ideal octahedron with vertices

$$0, 1, \infty, \exp(2\pi i\alpha), \exp(2\pi i\beta), \exp(2\pi i\kappa). \tag{36}$$

(c) Suppose that α, β, κ are real numbers that satisfy:

$$\alpha, \beta, \kappa \in [0,1] \qquad 0 \leq \alpha + \kappa \leq 1, \qquad 0 \leq \beta - \kappa \leq 1.$$

Then the following limit exists

$$r_-(\alpha, \beta, \kappa) := \lim_{n \to \infty} \frac{1}{n} |\mathrm{ev}_n(R_-(n; \lfloor n\alpha \rfloor, \lfloor n\beta \rfloor, \lfloor n\kappa \rfloor))|.$$

and is equal to

$$r_-(\alpha, \beta, \kappa) = r_+(\beta, \alpha, \kappa). \tag{37}$$

Proof. (a) Observe that

$$|\mathrm{ev}_n(\{j\})| = |\mathrm{ev}_n(\{n-j\})| = 2\sin(j\pi/n), \quad \text{and } |\mathrm{ev}_n(\{n-1\}!)| = \prod_{j=1}^{n-1} 2\sin(j\pi/n) = n.$$

From these, we have that

$$|\mathrm{ev}_n(\{j\}!)| = |\mathrm{ev}_n(\{n-1-j\}!)|. \tag{38}$$

Using (26) and then (38), we have

$$\begin{aligned} |\mathrm{ev}_n(R_+(n; a, b, k))| &= \frac{|\mathrm{ev}_n(\{b+k\}!)|\, |\mathrm{ev}_n(\{n-1+k-a\}!)|}{|\mathrm{ev}_n(\{b\}!)|\, |\mathrm{ev}_n(\{k\}!)|\, |\mathrm{ev}_n(\{n-1-a\}!)|} \\ &= \frac{|\mathrm{ev}_n(\{b+k\}!)|\, |\mathrm{ev}_n(\{a\}!)|}{|\mathrm{ev}_n(\{b\}!)|\, |\mathrm{ev}_n(\{k\}!)|\, |\mathrm{ev}_n(\{a-k\}!)|} \end{aligned} \tag{39}$$

Proposition 8.2 concludes the proof of (a).

(b) was pointed out to us D. Thurston. An ideal octahedron has eight six vertices, which modulo the isometries of hyperbolic space can be taken to be given by (36). It is easy to see that we can triangulate an ideal octahedron into five ideal tetrahedra. Their signed sum of volumes is given by the right hand side of (35).

(c) is analogous to (a). □

8.3. **The maximum of the growth rate of the R-matrix.** In this section we determine the maximum of $r_\pm(\alpha, \beta, \kappa)$.

Proposition 8.6. (a) With α, β, κ satisfying (33), $r_+(\alpha, \beta, \kappa)$ achieves maximum when $\alpha = 3/4, \beta = 1/4$, and $\kappa = 1/2$. Moreover

$$r_+(3/4, 1/4, 1/2) = \frac{v_8}{2\pi},$$

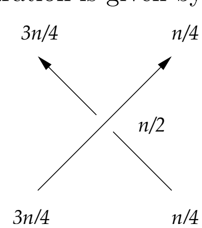
where

$$v_8 = 8\Lambda(\pi/4) \approx 3.66386237670887606 02\ldots$$

is the volume of the regular hyperbolic ideal octahedron.

(b) Similarly, $r_-(\alpha, \beta, \kappa)$ reaches maximum when $\alpha = 1/4, \beta = 3/4$, and $k = 1/2$; and its maximum value is the same as that of $r_+(\alpha, \beta, \kappa)$.

Thus, asymptotically, the winning configuration is given by:



Proof. It is enough to consider the case of r_+. The result for r_- follows from (37).

There are three proofs of the proposition, each useful in its own context.

The first proof is as follows. Let $\delta = \alpha - \kappa$, we have

$$r_+(\alpha, \beta, \kappa) = -\Lambda(\pi(\beta+\kappa)) + \Lambda(\pi\beta) + \Lambda(\pi\kappa) - \Lambda(\pi(\delta+\kappa)) + \Lambda(\pi(\delta)).$$

with domain $0 \leq \beta, \delta, \kappa$, and $\beta + \kappa \leq 1, \delta + \kappa \leq 1$. Note that symmetry between β and δ.

Using $\Lambda'(x) = -\log(2\sin x)$ for $0 < x < \pi$, one can easily show that the function $-\Lambda(\pi(\beta+\kappa)) + \Lambda(\pi\beta)$, for a fix $\kappa \in [0,1]$, achieves maximum at $\beta = (1-\kappa)/2$. It follows that the maximum of $r_+(\alpha, \beta, \kappa)$ is the same as the maximum of

$$g(\kappa) := 2(-\Lambda(\pi(\beta+\kappa)) + \Lambda(\pi\beta)) + \Lambda(\pi\kappa),$$

with $\beta = (1-\kappa)/2$. The domain for g is $\kappa \in [0,1]$. Using the derivative of g it is easy to show that g achieves maximum when $\kappa = 1/2$. In this case $\alpha = 3/4, \beta = 1/4$.

The second proof uses part (b) of Proposition 8.5 and the fact that the volume of an ideal octahedron is maximized at a regular ideal octahedron; see [Ra].

The third proof uses ideas from *angle structures*.

Since the domain (33) is compact and r_+ (given by (35)) is continuous, the maximum exists. Suppose that the maximum is at the interior (we will leave the other case to the reader). To find the critical points in the interior, let us set

$$z_\alpha = e^{2\pi i\alpha}, \qquad z_\beta = e^{2\pi i\beta}, \qquad z_\kappa = e^{2\pi i\kappa}.$$

Using the derivate of the Lobachevsky function

$$\Lambda'(x) = \log|e^{2ix} - 1|,$$

it follows that

$$\begin{aligned}
\frac{\partial r_+}{\partial \alpha} &= \log|z_\alpha - 1| - \log|z_\alpha z_\kappa^{-1} - 1| = 0 \\
\frac{\partial r_+}{\partial \beta} &= \log|z_\beta z_\kappa - 1| - \log|z_\beta - 1| = 0 \\
\frac{\partial r_+}{\partial \kappa} &= \log|z_\beta z_\kappa - 1| - \log|z_\kappa - 1| - \log|z_\alpha z_\kappa^{-1} - 1| = 0.
\end{aligned}$$

Thus, the critical points in the interior are given by the solutions of:

$$\begin{aligned}
|z_\alpha - 1| &= |z_\alpha z_\kappa^{-1} - 1| \\
|z_\beta z_\kappa - 1| &= |z_\beta - 1| \\
|z_\beta z_\kappa - 1| &= |z_\kappa - 1||z_\alpha z_\kappa^{-1} - 1|.
\end{aligned}$$

Using Lemma 8.7 below, and the fact that $z_\alpha, z_\beta, z_\kappa \neq 1$, the first two equations imply that

$$z_\alpha = z_\alpha^{-1} z_\kappa \qquad z_\beta z_\kappa = z_\beta^{-1}$$

Thus, $z_\kappa = z_\alpha^2 = z_\beta^2$ and $z_\alpha = \pm z_\beta$. Plugging in the third equation gives $(z_\alpha, z_\beta, z_\kappa) = (-i, i, -1)$, i.e., $(\alpha, \beta, \kappa) = (3/4, 1/4, 1/2)$. Since

$$\Lambda(\pi/2) = 0, \qquad \Lambda(3\pi/4) = -\Lambda(\pi/4),$$

it follows that

$$r_+(3/4, 1/4, 1/2) = \frac{4}{\pi}\Lambda(\pi/4) = \frac{v_8}{2\pi}.$$

The result follows. □

Lemma 8.7. *If z, w are complex numbers that satisfy $|z| = |w| = 1$ and $|1-z| = |1-w|$, then $z = w^{\pm}$.*

Proof. Let us define

$$C_{u_0,r} := \{u \in \mathbb{C} \mid |u - u_0| = r > 0\}.$$

Then $C_{u_0,r}$ is a circle with center u_0 and radius r. Fixing w, it follows that $z \in C_{0,1} \cap C_{1,|1-w|}$. The intersection of two circles is two points, and since w and $w^{-1} = \bar{w}$ both lie in the intersection, the result follows. □

8.4. **The maximum of the R-matrices at roots of unity.** Proposition 8.6 gives the maximum of the growth rate of $\mathrm{ev}_n(R_+(n; a, b, k))$, as $n \to \infty$. The following proposition gives the maximum of $\mathrm{ev}_n(R_+(n; a, b, k))$, for a fixed n.

Proposition 8.8. *The value of* $|\mathrm{ev}_n(R_+(n; a, b, k))|$ *achieves maximum at* $a = \lfloor(3n-3)/4\rfloor$, $b = \lfloor(n-1)/4\rfloor$, *and* $k = a - b$. *The value of* $|\mathrm{ev}_n(R_-(n; a, b, k))|$ *achieves maximum at* $a = \lfloor(n-1)/4\rfloor$, $b = \lfloor(3n-3)/4\rfloor$, *and* $k = b - a$. *The maximum values of* $|\mathrm{ev}_n(R_+(n; a, b, k))|$ *is the same as that of* $|\mathrm{ev}_n(R_-(n; a, b, k))|$.

Note that for these optimal values in the R_+ case, $|a - 3n/4| \le 1$, $|b - n/4| \le 1$ and $|k - n/2| \le 1$. The proof of this proposition will be given in the Appendix.

From Corollary 8.4 and Proposition 8.8, we have the following.

Corollary 8.9. *The growth rate of the maximum of* $|\mathrm{ev}_n(R_+(n; a, b, k))|$ *is given by*

$$\lim_{n\to\infty} \frac{\max_{a,b,k} |\mathrm{ev}_n(R_\pm(n; a, b, k))|}{n} = \frac{v_8}{2\pi}.$$

8.5. **Proof of Theorem 9.** Recall that by (28), the colored Jones function is the sum of n^c summands. Each summand $F(n, \mathbf{k})$ is the product of R-matrices (which are weights of crossing points) and weights of extreme points (which have absolute value 1). There are $c + 2$ crossing points, but the weights of the two boundary crossing have absolute value 1. Hence

$$|J_{K,n}(e^{2\pi i/n})| \le n^c \left(\max_{a,b,k} |\mathrm{ev}_n(R_\pm(n; a, b, k))|\right)^c.$$

From the growth rate of $\max_{a,b,k} |\mathrm{ev}_n(R_\pm(n; a, b, k))|$ given by Corollary 8.9 we get the theorem.

9. The generalized volume conjecture near $\alpha = 2\pi i$

In this section we will prove Theorems 6 and 7 which are concerned with the Generalized Volume Conjecture near $2\pi i$. Our proofs use crucially the well-known *symmetry principle*, see [KM, Le1]: Suppose m, m' and n are positive integers with $m \equiv \pm m' \bmod n$, then

$$J_{K,m}(e^{2\pi i/n}) = J_{K,m'}(e^{2\pi i/n}). \tag{40}$$

Note that this fact is also a consequence of the existence of the cyclotomic expansion. However, the case of higher rank Lie algebra requires results from canonical basis theory, see [Le1].

Proof. (of Theorem 6) The symmetry principle implies that for all $n > m > 0$, we have:

$$J_{K,n\pm m}(e^{2\pi i/n}) = J_{K,m}(e^{2\pi i/n})$$

which implies that

$$\lim_{n\to\infty} J_{K,n\pm m}(e^{2\pi i/n}) = \lim_{n\to\infty} J_{K,m}(e^{2\pi i/n}) = J_{K,m}(1) = 1,$$

from which Theorem 6 follows easily. □

Proof. (of Theorem 7) Fix a knot K and consider the neighborhood U_K of 0 as in Theorem 2. Define $V_K = 1 + U_K$.

Let us suppose that $\alpha/(2\pi i) \in V_K$ is a rational number not equal to 1. Assume that $\alpha = 2\pi i p/m$ with p, m unequal coprime positive integers. Let $N = np$. Then, the symmetry principle implies that

$$\begin{aligned} f_{K,N}(\alpha) &= J_{K,N}(e^{\alpha/N}) \\ &= J_{K,np}(e^{2\pi i/(nm)}) \\ &= J_{K,n|p-m|}(e^{2\pi i/(nm)}). \end{aligned}$$

Since $n|p-m|/(nm) = |p/m-1| \in U_K$. Theorem 2 implies that

$$\lim_{n\to\infty} J_{K,n|p-m|}(e^{2\pi i/(nm)}) = \frac{1}{\Delta(e^{2\pi i(|p/m-1|})}.$$

In other words,

$$\lim_{n\to\infty} f_{K,np}(\alpha) = \frac{1}{\Delta(e^{2\pi i(|p/m-1|})}$$

is bounded. The result follows. □

10. The q-holonomic point of view

10.1. **Bounds on l^1-norm of q-holonomic functions.** The main result of [GL] is that for every knot K, the functions J_K and H_K are q-holonomic. Recall that a sequence $f : \mathbb{N} \longrightarrow \mathbb{Q}(q)$ is *q-holonomic* if satisfies a q-linear difference equation. In other words, there exists a natural number d and polynomial $a_j(u,v) \in \mathbb{Q}[u,v]$ for $j = 0, \dots, d$ with $a_d \neq 0$ such that for all $n \in \mathbb{N}$ we have:

$$\sum_{j=0}^{d} a_j(q^n, q) f_{n+j}(q) = 0.$$

In this section we observe that q-holonomic functions satisfy *a priori* upper bounds on their degrees and (under an integrality assumption) on their l^1-norm. As a simple corollary, we obtain another proof of the bounds in Proposition 6.1.

Definition 10.1. We say that a sequence $f : \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}]$ is *q-integral holonomic* if it satisfies an *q-difference equation* as above with $a_d = 1$.

Question 1. Is it true that J_K and C_K are q-integral holonomic for every knot K?

For a partial answer, see [GS].

Theorem 15. (a) *If $f : \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}]$ is q-holonomic, then for all n we have:*

$$\deg_+(f_n) = O(n^2) \qquad \text{and} \qquad \deg_-(f_n) = O(n^2).$$

(a) *If f is q-integral holonomic, then for all n we have:*

$$||f_n||_1 \leq C^n$$

for some constant C. In particular,

$$\limsup_{n\to\infty} \frac{\log |f_n(e^{\alpha/n})|}{n} \leq C_\alpha$$

for all $\alpha \in \mathbb{C}$.

In other words, integral q-holonomic functions grow at most exponentially.

Proof. We have

$$a_d(q^n, q) f_{n+d}(q) = -a_{d-1}(q^n, q) f_{n+d-1}(q) - \dots a_0(q^n, q) f_n(q).$$

Choose C' so that

- $\deg_+ a_j(q^n, q) \leq 2C'(n+d)$ for all $j = 0, \dots, d-1$, and
- $\deg_+ f(n) \leq C'(n+1)^2$ for $n = 0, \dots, d-1$.

We will prove by induction on n that $\deg_+ f(n) \leq C'(n+1)^2$. By assumption, it is true for $n = 0, \dots, d-1$. Then, by induction we have:

$$
\begin{aligned}
\deg_+ f_{n+d}(q) &\leq \deg_+ a_d(q^n, q) + \deg_+ f_{n+d}(q) \\
&= \deg_+ (a_d(q^n, q) f_{n+d}(q)) \\
&\leq \max_{0 \leq j < d} \operatorname{maxdeg}(a_j(q^n, q) f_{n+j}(q)) \\
&= \max_{0 \leq j < d} 2C'(n+j+1) + C'(n+j+1)^2 \\
&= 2C'(n+d) + C'(n+d)^2 \\
&< C'(n+d+1)^2.
\end{aligned}
$$

The second claim in (a) follows similarly.

For (b), let $c_j = ||a_j(Q, q)||_1$ for $j = 0, \dots, d-1$, and choose C so that

- $C^d \leq c_{d-1} C^{d-1} + \dots + c_0 C^0$, and
- $||f_n(q)||_1 \leq C^n$ for $n = 0, \dots, d-1$.

Then, it is easy to see by induction that (b) holds for all n. □

Remark 10.2. It is easy to see that the bounds of Theorem 15 are sharp. For example, consider the sequence $f_n(q) = (1+q)(1+q^2)\dots(1+q^n)$.

Theorem 15 gives an alternative proof of Proposition 6.1 (and Theorem 13 if Question 1 has a positive answer). However, the explicit upper bounds in terms of the number of crossings cannot be obtained from Theorem 15, unless we know something more about the q-difference equation of the colored Jones function.

10.2. **Bounds for higher rank groups.** In [GL], we considered the colored Jones function

$$J_{\mathfrak{g},K} : \Lambda_w \longrightarrow \mathbb{Z}[q^{\pm}]$$

of a knot K, where $\mathfrak{g}$ is a *simple Lie algebra* with *weight lattice* Λ_w. In the above reference, the authors proved that $J_{\mathfrak{g},K}$ is a q-holonomic function, at least when $\mathfrak{g}$ is not G_2. For $\mathfrak{g} = \mathfrak{sl}_2$, $J_{\mathfrak{sl}_2,K}$ is the colored Jones function J_K discussed earlier.

In [GL] , the authors gave state-sum formulas for $J_{\mathfrak{g},K}$ similar to (28) where the summand takes values in $\mathbb{Z}[q^{\pm 1/D}]$, where D is the size of the center of $\mathfrak{g}$.

The methods of the present paper give an upper bound for the growth-rate of the $\mathfrak{g}$-colored Jones function. More precisely, we have:

Theorem 16. *For every simple Lie algebra $\mathfrak{g}$ (other than G_2) and every $\alpha \in \mathbb{C}$, and every $\lambda \in \Lambda_w$, there exists a constant $C_{\mathfrak{g},\alpha,\lambda}$ such that for every knot with $c+2$ crossings, we have:*

$$\limsup_{n \to \infty} \frac{\log |J_{\mathfrak{g},K,n\lambda}(e^{\alpha/n})|}{n} \leq C_{\mathfrak{g},\alpha,\lambda} c.$$

The details of the above theorem will be explained in a subsequent publication.

11. Some physics

11.1. **A small dose of physics.** One does not need to know the relation of the colored Jones function and quantum field theory in order to understand the statement and proof of Theorem 4. Nevertheless, we want to add some philosophical comments, for the benefit of the willing reader. According to Witten (see [Wi]), the Jones polynomial $J_{K,n}$ can be expressed by a partition function of a topological quantum field theory in 3 dimensions—a gauge theory with Chern-Simons Lagrangian. The stationary points of the Lagrangian correspond to $SU(2)$-flat connections on an ambient manifold, and the observables are knots, colored by the n-dimensional irreducible representation of $SU(2)$. In case of a knot in S^3, there is only one ambient flat connection, and the corresponding perturbation theory is a formal power series in $h = \log q$.

Rozansky exploited a cut-and-paste property of the Chern-Simons path integral and considered perturbation theory of the knot complement, along an abelian flat connection with monodromy given by (7). In fact, Rozansky calls such an expansion the $U(1)$-*RCC connection contribution* to the Chern-Simons path integral, where RCC stands for *reducible connection contribution*, and $U(1)$ stands for the fact that the flat $SU(2)$ connections are actually $U(1)$-valued abelian connections. Formal properties of such a perturbative expansion, enabled Rozansky to deduce (in physics terms) the loop expansion of the colored Jones function. In a later publication, Rozansky proved the existence of the loop expansion using an explicit state-sum description of the colored Jones function.

Of course, perturbation theory means studying formal power series that rarely converge. Perturbation theory at the trivial flat connection in a knot complement converges, as it resums to a Laurent polynomial in e^h; namely the nth colored Jones polynomial. The volume conjecture for small complex angles is precisely the statement that perturbation theory for abelian flat connections (near the trivial one) does converge.

At the moment, there is no physics (or otherwise) formulation of perturbation theory of the Chern-Simons path integral along a discrete and faithful $\mathrm{SL}_2(\mathbb{C})$ representation. Nor is there an adequate explanation of the relation between $SU(2)$ gauge theory (valid near $\alpha = 0$) and a complexified $\mathrm{SL}_2(\mathbb{C})$ gauge theory, valid near $\alpha = 2\pi i$. These are important and tantalizing questions, with no answers at present.

11.2. **The WKB method.** Since we are discussing physics interpretations of Theorem 4 let us make some more comments. Obviously, when the angle α is sufficiently big, the asymptotic expansion of Equation (6) may break down. For example, when e^α is a complex root of the Alexander polynomial, then the right hand side of (6) does not make sense, even to leading order. In fact, when α is near $2\pi i$, then the solutions are expected to grow exponentially, and not polynomially, according to the Volume Conjecture.

The breakdown and change of rate of asymptotics is a well-documented phenomenon well-known in physics, associated with WKB analysis, after Wentzel-Krammer-Brillouin; see for example [O]. In fact, one may obtain an independent proof of Theorem 4 using *WKB analysis*, that is, the study of asymptotics of solutions of difference equations with a small parameter. The key idea is that the sequence of colored Jones functions is a solution of a linear q-difference equation, as was established in [GL]. A discussion on WKB analysis of q-difference equations was given by Geronimo and the first author in [GG].

The WKB analysis can, in particular, determine *small exponential corrections* of the form $e^{-c_\alpha n}$ to the asymptotic expansion of Theorem 4, where c_α depends on α, with $\mathrm{Re}(c_\alpha) < 0$ for α sufficiently small. These exciting small exponential corrections cannot be captured by classical asymptotic analysis (since they vanish to all orders in n), but they are important and dominant (i.e., $\mathrm{Re}(c_\alpha) > 0$) when α is near $2\pi i$, according to the volume conjecture. Understanding the change of sign of $\mathrm{Re}(c_\alpha)$ past certain so-called Stokes directions is an important question that WKB addresses.

We will not elaborate or use the WKB analysis in the present paper. Let us only mention that the loop expansion of the colored Jones function can be interpreted as WKB asymptotics on a q-difference equation satisfied by the colored Jones function.

Appendix A. The volume conjecture for the Borromean rings

It is well-known that the complement of the Borromean rings B can be geometrically identified by gluing two regular ideal octahedra, see [Th]. As a result, the volume $\mathrm{vol}(S^3 - B)$ of $S^3 - B$ is equal to $2v_8$.

Suppose L is a k-component framed link, and $n_1, \dots, n_k$ are positive integers. The colored Jones polynomial $\tilde{J}_L(n_1, \dots, n_k) \in \mathbb{Z}[q^{\pm 1/4}]$ is the sl_2-quantum invariant of the link whose components are colored by sl_2-modules of dimensions $n_1, \dots, n_k$, see [RT, Tu]. The normalization is chosen so that for the unknot, $\tilde{J}_L(n) = [n]$. Define

$$J_{L,n}(q) := \frac{J_L(n, n, \dots, n)}{[n]}.$$

The next theorem confirms the volume conjecture for the Borromean rings.

Theorem 17. *Let B be the Borromean rings, then*

$$\lim_{n\to\infty} \frac{\log |J_{B,n}(e^{2\pi i/n})|}{n} = \frac{1}{2\pi}\mathrm{vol}(S^3 - B).$$

Before giving the proof let us introduce some notation. Fix a positive integer n. For an integer j and a positive integer k let $x_j = 2\sin(j\pi/n)$.

Then, see (39),

$$x_j = x_{n-j} = -x_{n+j}, \quad \text{and} \quad \prod_{j=1}^{n-1} x_j = n, \quad \text{hence}$$

(41) $$z_k = n/z_{n-1-k} \quad \text{for } 1 \leq k \leq n-1, \qquad \text{where} \quad z_k := \prod_{j=1}^{k} x_j.$$

Proof. Using Habiro's formula for $\tilde{J}_L$ of the Borromean ring [H2], one has

$$J_{B,n}(q) = \sum_{l=0}^{n-1} (-1)^l \, \frac{\{n\}^2 \left(\prod_{j=1}^{l} \{n+j\}\{n-j\}\right)^3}{\left(\prod_{j=l+1}^{2l+1} \{j\}\right)^2}.$$

When $q^{1/2} = e^{i\pi/n}$, on has $\{j\} = 2i \sin \frac{j\pi}{n}$, which is 0 exactly when j is divisible by n. Hence if $2l+1 < n$, then the denominator of the term in the above sum is never 0, while the numerator is 0, since it has 2 factors $\{n\}$. On the other hand, if $2l+1 > n$, then the denominator has 2 factors $\{n\}$, which would cancel with the 2 same factors of the numerator. Hence at $q^{1/2} = e^{i\pi/n}$ one can assume that $2l+1 \geq n$, or $l > n/2 - 1$:

$$J_{B,n}(e^{2\pi i/n}) = \sum_{n > l > n/2-1} (-1)^l \operatorname{ev}_n \frac{\left(\prod_{j=1}^{l} \{n+j\}\{n-j\}\right)^3}{\left(\prod_{j=l+1}^{n-1} \{j\} \prod_{j=n+1}^{2l+1} \{j\}\right)^2}.$$

One important observation is that each summand on the right hand side is a positive real number:

$$J_{B,n}(e^{2\pi i/n}) = \sum_{n > l > n/2-1} \frac{(z_l)^8}{(z_{n-1})^2 \, (z_{2l+1-n})^2},$$

Using $z_{n-1} = n$, and $z_l = n/z_{n-1-l}$, we have

$$J_{B,n}(e^{2\pi i/n}) = \sum_{n > l > n/2-1} \left(\frac{(z_l)^2}{(z_{n-1-l})^2 \, z_{2l+1-n}} \right)^2.$$

By (44) below, we have:

$$\frac{(z_l)^2}{(z_{n-1-l})^2 \, z_{2l+1-n}} = |\operatorname{ev}_n(R_+(n; a, b, k))|^2$$

where $a = l$, $b = n-1-l$ and $k = 2l+1-n$. Note that when $l = \lfloor (3n-3)/4 \rfloor$, then a, b, k are exactly the ones that make $|\operatorname{ev}_n(R(n; a, b, k))|$ maximum, according to Proposition 8.8. There are less than n summands, hence

$$\max_{a,b,k} |\operatorname{ev}_n(R(n; a, b, k))|^2 < J_{B,n}(e^{2\pi i/n}) < n \max \max_{a,b,k} |\operatorname{ev}_n(R(n; a, b, k))|^2$$

It follows that

$$2\pi \lim_{n\to\infty} \frac{\log |J_{B,n}(e^{2\pi i/n})|}{n} = \lim_{n\to\infty} \frac{\log \max_{a,b,k} |\operatorname{ev}_n(R(n; a, b, k))|^2}{n},$$

which is equal to $2v_8$, according to Corollary 8.9. $\square$

Appendix B. The volume conjecture for torus knots

Let $T(a,b)$ denote the (a,b)-torus knot for co-prime integers a,b $(a,b>1)$. Although $T(a,b)$ is not a hyperbolic knot, the *volume function* $\mathrm{vol}(\rho)$ of SL_2-representations can be defined, see [CCGLS]. Let us discuss this first. The derivative of the volume function, or the volume form, depends only on its pull-back on the deformation variety, which is the set of 0 of the A-polynomial $A_K(l,m)$. For torus knots, the A-polynomial is either of the form $l \pm m^c$ for some integer c, or the product of 2 factors of that forms. Using explicit formula of the volume form, it is easy to check that the volume form on any such factor is equal to 0.

Thus the volume function on the deformation variety must be constant. Because the Gromov norm of the knot complement is 0, one should define $V(l,m)=0$ for every l,m. Then the generalized volume conjecture can be proved easily in this case:

Proposition B.1. *For the torus knot $K=T(a,b)$ and for any real number α, one has*

$$\lim_{n\to\infty} \frac{\log |J_{K,n}(e^{2\pi i\alpha/n})|}{n} = 0. \tag{42}$$

Proof. There are two cases: α is an integer, or α is not. The first case is actually much more difficult, this is the usual volume conjecture and it has been proved by Kashaev and Tirkkonen in [KT]. Let us consider the easier case, when α is not an integer.

Note that if $q^{1/2}=\exp(\pi i\alpha/n)$ and α is not an integer then $\{n\} = 2i\sin \pi\alpha/n \neq 0$.

The colored Jones polynomial was calculated by Morton in [Mo]:

$$J_{K,n}(q) = \frac{v^{-ab(n^2-1)/2}}{v^n - v^{-n}} \sum_{k=1-n,\, k+n\equiv 1 \pmod 2}^{n-1} (v^{2abk^2+2ka+2kb+1} - v^{2abk^2+2ka-2kb-1}). \tag{43}$$

The sum contains $n+1$ terms, each by absolute value is less than or equal to 2. Hence

$$|J_{K,n}(e^{2\pi i\alpha/n})| < \frac{2(n+1)}{\sin(\pi\alpha/n)}.$$

Thus, the limsup is less than or equal to 0. The argument in Section 4 shows that the lim inf is greater than or equal to 0. The result follows. □

Remark B.2. In [M2], H. Murakami discusses the Generalized Volume Conjecture for the torus knots and angles α with nonzero imaginary part.

Appendix C. Proof of Proposition 8.8

Recall that $x_j = 2\sin(j\pi/n)$.

Lemma C.1. (a) *The x_j, as a function of j, is increasing for $j\in[0,n/2]$ and decreasing for $j\in[n/2,n]$. In particular, for $j \le l \le n-j$ we have $x_j \le x_l$.*

(b) *For every $1\le j\le n-1$, one has $2 \ge x_j$. For $n/4 \le j \le 3n/4$, one has $2 \le (x_j)^2$. For $1 \le j \le n/4$, one has $(x_j)^2 \le x_{2j}$.*

Proof. Only the last statement of (b) is not trivial, and here is a proof. For $1 \le j \le n/4$, one has

$$\sin(j\pi/n) \le \cos(j\pi/n).$$

Multiply both side by $4\sin(j\pi/n)$, using the identity $\sin(2u) = 2\sin u\cos u$, we get $(x_j)^2 \le x_{2j}$. □

Lemma C.2. *For a fixed k, $1\le k\le n-1$, the value of $y_b := \prod_{j=b+1}^{b+k} x_j$ achieves maximum at $b = \lfloor (n-k)/2 \rfloor$.*

Proof. We will prove that if $b < \lfloor (n-k)/2 \rfloor$, then $y_b \le y_{b+1}$, while if $b > \lfloor (n-k)/2 \rfloor$ then $y_b \le y_{b-1}$. This will prove the lemma.

Suppose $b < \lfloor (n-k)/2 \rfloor$. Then $b \le \lfloor (n-k)/2 \rfloor - 1 \le (n-k)/2 - 1$. It follows that $(b+1) \le b+k+1 \le n-(b+1)$. From Lemma C.1(a) we get $x_{b+1} \le x_{b+k+1}$. Hence $y_{b+1}/y_b = x_{b+k+1}/x_{b+1} \ge 1$, or $y_{b+1} \ge y_b$.

Suppose now $b \ge 1 + \lfloor (n-k)/2 \rfloor$. If $b \ge n/2$, then $x_b \ge x_{b+k}$ since y_j is deceasing on $[n/2, n]$. If $b < n/2$, then from $b \ge 1 + \lfloor (n-k)/2 \rfloor$ one can easily show that $b + k \ge n - b \ge n/2$. Hence we also have $x_b = x_{n-b} \ge x_{b+k}$. Thus $y_{b-1}/y_b = x_b/x_{b+k} \ge 1$. □

Recall that $z_k = \prod_{j=1}^k x_j$. Using (39), with $|\mathrm{ev}_n(\{j\})| = x_j$, we have

$$|\mathrm{ev}_n R_+(n; a, b, k)| = \frac{z_{b+k} z_a}{z_b z_k z_{a-k}} = \frac{z_{b+k} z_{d+k}}{z_b z_k z_d} \quad \text{with } d = a - k. \tag{44}$$

Since $z_{b+k}/z_b = y_b$ in Lemma C.2, by this lemma, both z_{b+k}/z_b and z_{d+k}/z_d achieve maximum when $b = d = \lfloor (n-k)/2 \rfloor$. To find maximum of $|R_+|$ we need to find the maximum of

$$s(k) := \frac{(z_{b+k})^2}{(z_b)^2 z_k},$$

with $b = \lfloor (n-k)/2 \rfloor$. It is easy to check that

$$\frac{s(k+1)}{s(k)} = \frac{(x_b)^2}{x_{k+1}}.$$

Lemma C.3. *The function $s(k)$ achieves maximum at an integer k in the interval $(\frac{n-2}{2}, \frac{n+3}{2})$.*

Proof. We we prove that if $k < \frac{n}{2} - 1$ then $s(k+1) \ge s(k)$, and if $k > \frac{n+1}{2}$, then $s(k+1) \le s(k)$. This will prove the lemma.

Suppose $k \le n/2 - 1$. Note that $b = \lfloor (n-k)/2 \rfloor \ge$ is always greater or equal to $(n-k-1)/2$, which, in turn, is greater than or equal to $n/4$. Hence $b \ge n/4$. By Lemma C.1(b), we have $(x_b)^2 \ge 2 \ge x_{k+1}$. Hence $s(k+1)/s(k) \ge 1$.

Now suppose $k > n/2 + 1/2$. Consider 2 cases: $n-k$ is odd or even. Suppose $n-k$ is odd. Then $b = (n-k-1)/2 < n/4$. Hence by Lemma C.1(b), we have

$$(x_b)^2 < x_{2b} = x_{n-k-1} = x_{k+1}.$$

It follows that $\frac{s(k+1)}{s(k)} = \frac{(x_b)^2}{x_{k+1}} < 1$.

Suppose $n-k$ is even. Then $b = (n-k)/2$, and we have $b \le (n-1)/4$. Note that $x_{k+1} = x_{n-1-1} = x_{2b-1}$. In this case the fact that $(x_b)^2 < x_{2b-1}$ is the statement of Lemma C.5 below. □

There are only 2 values of k in the interval $((n-2)/2, (n+3)/2)$. For example, if $n = 4m$, then $k = 2m$ or $k = 2m+1$. Direct comparison show that the maximum in this case is when $k = 2m+1$, then $b = \lfloor (n-k)/2 \rfloor = m-1$, and $a = b+k = 3m$. Other cases when $n = 4m+r$, with $r = 1, 2, 3$, can be checked by comparing the values at the two possible k. The answer is given as in Proposition 8.8. This completes the proof of Proposition 8.8, modulo Lemma C.5 below.

Lemma C.4. *The function $g(x) = \sin x \sin(\frac{\pi}{4} - x)$ on the interval $[\varepsilon, \frac{\pi}{4} - \varepsilon]$, with $0 \le \varepsilon \le \frac{\pi}{8}$, achieves minimum at $x = \varepsilon$.*

Proof. One has $g'(x) = \sin(\frac{\pi}{4} - 2x)$, hence $g(x)$ is increasing on $[0, \frac{\pi}{8}]$ and decreasing on $[\frac{\pi}{8}, \frac{\pi}{4}]$. From here the result follows. □

Lemma C.5. *Let $u = \pi/n$, where $n > 40$ is an integer. Suppose $1 \le b \le (n-1)/4$. Then*

$$\sin(2b-1)u \ge 2\sin^2 bu.$$

Proof. One has

$$\sin(2b-1)u - 2\sin^2 bu = S_1 - S_2, \quad \text{where}$$
$$S_1 = \sin 2bu - 2\sin^2 bu$$
$$S_2 = \sin 2bu - \sin(2b-1)u.$$

Let $d = n/4 - b$. Then $bu = \pi/4 - du$, and $\cos bu = \sin(\pi/4 + du)$. Using $\sin 2bu = 2\sin bu \cos bu$, we have

$$\begin{aligned} S_1 &= 2\sin bu \times (\cos bu - \sin bu) \\ &= 2\sin bu \times (\sin(\pi/4+du) - \sin(\pi/4-du)) \\ &= 4\sin bu \times 2\cos(\pi/4)\sin du, \quad \text{by trigonometry identity} \end{aligned}$$

Thus

$$S_1 = 2\sqrt{2}\sin bu \sin du. \tag{45}$$

$$\begin{aligned} S_2 &= 2\sin\frac{u}{2}\cos(2b-\frac{1}{2})u, \quad \text{by trigonometry identity} \\ &= 2\sin\frac{u}{2}\sin(2d+\frac{1}{2})u, \quad \text{since } (2b-\frac{1}{2})u + (2d+\frac{1}{2})u = \frac{\pi}{2}. \end{aligned}$$

Consider two cases.

Case 1: $1 \le d$. By assumption $1 \le b$. Note that $du + bu = \pi/4$. From (45) and Lemma C.4 we see that S_1 reaches minimum when $d = 1$, i.e.

$$\begin{aligned} S_1 &\ge 2\sqrt{2}\sin(\frac{\pi}{4} - u)\sin u \\ &> \sqrt{2}\sin u, \quad \text{since } \frac{\pi}{4} - u > \frac{\pi}{6} \\ &> 2\sqrt{2}\sin\frac{u}{2}\cos\frac{u}{2} > 2\sin\frac{u}{2} \quad \text{since } u \le \pi/40 \text{ and } \sqrt{2}\cos u > 1 \end{aligned}$$

On the other hand,

$$S_2 = 2\sin\frac{u}{2}\sin(2d+\frac{1}{2})u < 2\sin\frac{u}{2}.$$

Thus $S_2 < S_1$.

Case 2: $1/4 \le d < 1$. Then $2d + 1/2 < 5/2$, and

$$S_2 < 2\sin\frac{u}{2}\sin\frac{5u}{2} = 4\sin\frac{u}{4}\cos\frac{u}{4}\sin\frac{5u}{2} < 4\sin\frac{u}{4}\sin\frac{5u}{2}.$$

Using $\sin\frac{5u}{2} < \frac{5u}{2} = 5\pi/2n < 1/4$, we have $S_2 < \sin\frac{u}{4}$.

On the other hand, Lemma C.4 shows that S_1 reaches minimum when $d = 1/4$, i.e.

$$S_1 \ge 2\sqrt{2}\sin(\frac{\pi}{4} - \frac{u}{4})\sin\frac{u}{4} > \sqrt{2}\sin\frac{u}{4}, \quad \text{since } \frac{\pi}{4} - \frac{u}{4} > \pi/6.$$

We also have $S_1 > S_2$. □

References

[Ag] I. Agol, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, preprint 1999 `math.GT/9906182`.

[ADST] ——, N. Dunfield, P. Storm and W. Thurston, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, preprint 2005, `math.DG/0506338`.

[An] G.E. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998.

[B-N] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423–472.

[B-NG] —— and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, Inventiones, **125** (1996) 103–133.

[CCGLS] D. Cooper, D, M. Culler, H. Gillet, D. Long and P. Shalen, *Plane curves associated to character varieties of* 3-*manifolds*, Invent. Math. **118** (1994) 47–84.

[Ga1] S. Garoufalidis, *Beads: from Lie algebras to Lie groups*, preprint `math.GT/0201056`.

[GR] —— and L. Rozansky, *The loop expansion of the Kontsevich integral, the null-move and S-equivalence*, Topology, **43** (2004) 1183–1210.

[GL] ——— and TTQ. Le, *The colored Jones function is q-holonomic* Geom. and Topology, **9** (2005) 1253–1293.
[GG] ——— and J. Geronimo, *Asymptotics of q-difference equations*, Contemporary Math. AMS **416** (2006) 83–114.
[GK] ——— and A. Kricker, *A rational noncommutative invariant of boundary links*, Geom. and Topology **8** (2004) 115–204.
[GS] ——— and X. Sun, to appear.
[Gu] S. Gukov, *Three-Dimensional Quantum Gravity, Chern-Simons Theory, and the A-Polynomial*, Comm. Math. Phys. 255 (2005), no. 3, 577–627.
[H1] K. Habiro, *On the quantum* $\mathfrak{sl}_2$ *invariants of knots and integral homology spheres*, Geom. Topol. Monogr. **4** (2002) 55–68.
[H2] ———, *On the colored Jones polynomials of some simple links*, Recent progress towards the volume conjecture (Japanese) (Kyoto, 2000) 34–43.
[Hi] E. Hille, *Analytic function theory*, vols I, II, Boston, Mass.-New York-Toronto 1962.
[J] V. Jones, *Hecke algebra representation of braid groups and link polynomials*, Annals Math. **126** (1987) p. 335–388.
[K] R. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Modern Phys. Lett. A **39** (1997) 269–275.
[KT] ——— and O. Tirkkonen, *A proof of the volume conjecture on torus knots*, J. Math. Sci. **115** (2003) 2033–2036.
[KM] R. and P. Melvin, *The* 3*-manifold invariants of Witten and Reshetikhin-Turaev for* $\mathrm{sl}(2, C)$, Invent. Math. **105** (1991), 473–545.
[La] M. Lackenby, *The volume of hyperbolic alternating link complements*, With an appendix by Ian Agol and Dylan Thurston. Proc. London Math. Soc. **88** (2004) 204–224.
[Le1] T. T. Q. Lê, *Integrality and symmetry of quantum link invariants*, Duke Math. J., **102** (2000) 273–306.
[Le2] T. Le, *The Colored Jones Polynomial and the A-Polynomial of Two-Bridge Knots*, Adv. Math. **207** (2006) 782–804.
[Mo] H. Morton, *The coloured Jones function and Alexander polynomial for torus knots*, Math. Proc. Cambridge Philos. Soc. **117** (1995) 129–135.
[MM] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001) 85–104.
[M1] ———, *Some limits of the colored Jones polynomials of the figure-eight knot*, Kyungpook Math. J. **44** (2004) 369–383.
[M2] ———, *Asymptotic behaviors of the colored Jones polynomials of a torus knot*, Internat. J. Math. **15** (2004) 547-555.
[O] F. Olver, *Asymptotics and special functions*, Reprint. AKP Classics. A K Peters, Ltd., Wellesley, MA, 1997.
[Ra] G.J. Ratcliffe, *Foundations of hyperbolic manifolds*, Second edition. Graduate Texts in Mathematics, **149** Springer-Verlag 2006.
[RT] N. Reshetikhin, V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990) 1–26.
[Ro] L. Rozansky, *The universal R-matrix, Burau representation, and the Melvin-Morton expansion of the colored Jones polynomial*, Adv. Math. **134** (1998) 1–31.
[Sch] J. Schiff, *Normal families*, Springer-Verlag, New York, 1993.
[Th] W. Thurston, *The geometry and topology of 3-manifolds*, 1979 notes, available from MSRI.
[Tu] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics 18, Walter de Gruyter, Berlin New York 1994.
[Wi] E.Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Physics. **121** (1989) 360–376.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA, http://www.math.gatech.edu/~stavros
E-mail address: stavros@math.gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA
E-mail address: letu@math.gatech.edu