CONCORDANCE AND 1-LOOP CLOVERS

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Abstract. We show that surgery on a connected clover (or clasper) with at least one loop preserves the concordance class of a knot. Surgery on a slightly more special class of clovers preserves invertible concordance. We also show that the converse is false. Similar results hold for clovers with at least two loops vs. \( S \)-equivalence.

1. Introduction

1.1. History. M. Goussarov and K. Habiro have independently studied links and 3-manifolds from the point of view of surgery on objects called \( Y \)-graphs, clasps or clovers, respectively by [Gu, H] and [GGP]. Following the notation of [GGP], given a pair \((M, K)\) consisting of a knot \( K \) in an integral homology 3-sphere \( M \), and a clover \( G \subset M - K \), surgery on the framed link associated to \( G \) produces a new pair \((M, K)_G\). Thus, by specifying a class of clovers \( \mathcal{C} \) we can define an equivalence relation (also denoted by \( \mathcal{C} \)) on the set \( \mathcal{K} \) of knots in integral homology 3-spheres and sometimes on its subset \( \mathcal{K} \) of knots in \( S^3 \).

It is often the case that for certain classes of clovers \( \mathcal{C} \), the equivalence relation is related to some natural topological equivalence relation. In this paper we will be particularly interested in concordance (in the smooth category) but will also discuss \( S \)-equivalence.

We begin by discussing some known facts. Using the terminology of [GGP], let \( \Delta^\Delta \) denote the class of clovers \( G \subset S^3 - K \) of degree 1 (that is, the class of \( Y \)-graphs) whose leaves form a 0-framed unlink which bounds disks disjoint from \( G \) that intersect \( K \) geometrically twice and algebraically zero times. Surgery on such clovers was called a double \( \Delta \) move by Naik-Stanford, who showed that

**Theorem 1.** [NS] \( \Delta^\Delta \) coincides with \( S \)-equivalence on \( \mathcal{K} \).

Relaxing the above condition, let \( \mathcal{C}^{\text{loop}} \) denote the class of clovers \( G \subset M - K \) whose leaves have zero linking number with \( K \). Surgery on such clovers was called a loop move by G.-Rozansky who showed that

**Theorem 2.** [GR] \( \mathcal{C}^{\text{loop}} \) coincides with \( S \)-equivalence on \( \mathcal{K}M \).

Let us make the following definition. If \( G \) is a clover in \( M - K \) and \( \mathcal{L} \) a set of leaves of \( G \), we say \( \mathcal{L} \) is simple if the elements of \( \mathcal{L} \) bound disks in \( M \) each of which intersects \( K \) exactly once but whose interiors otherwise are disjoint from \( K, G \) and each other. Consider now for every non-negative integer \( n \), the class \( \mathcal{C}^n \) of clovers \( G \subset S^3 - K \) whose entire set of leaves is simple, and such that each connected component of \( G \) is a graph with at least \( n \) loops (i.e., whose first betti number is at least \( n \)). Kricker and Murakami-Ohtsuki showed that

**Theorem 3.** [Kr, MO] \( \mathcal{C}^\mathcal{L} \) implies \( S \)-equivalence on \( \mathcal{K} \).

In fact, if we let \( \mathcal{C}^{\mathcal{L}^\mathcal{P}} \) denote the class of clovers \( G \) such that each component of \( G \) has at least one internal trivalent vertex, and \( G \) has a simple set of leaves containing one leaf from each component, then it is not hard to check that \( \mathcal{C}^\mathcal{L} \subset \mathcal{C}^{\mathcal{L}^\mathcal{P}} \) and [Kr, MO] actually proved that \( \mathcal{C}^{\mathcal{L}^\mathcal{P}} \) implies \( S \)-equivalence. Combining this with a recent result of Conant-Teichner [CT] we actually have:

**Theorem 4.** [CT] \( \mathcal{C}^{\mathcal{L}^\mathcal{P}} \) coincides with \( S \)-equivalence on \( \mathcal{K} \).
1.2. Statement of the results. In the present paper we will prove the following results.

**Theorem 5.** $c^1$ implies concordance on $K$.

An different proof of Theorem 5 has been obtained by Conant-Teichner [CT] relying on the notion of grope cobordism. This result was also announced by the first author in [Le2], where an analogous statement was proved, and our proof will follow the lines of that argument. The result was also known to Habiro, according to private communication.

A slight refinement of the class $c^1$ relates to a classical refinement of concordance known as invertible concordance. Recall that a knot in $S^3$ is called double-slice if it can be exhibited as the intersection of a 3-dimensional hyperplane in $\mathbb{R}^4$ with an unknotted imbedding of $S^2$ in $\mathbb{R}^4$; see e.g. [Su]. Such knots are obviously slice, and it is shown in [Su] that, for any knot $K$, the connected sum $K \# (-K)$ is double-slice, where $-K$ denotes the mirror image of $K$. On the other hand the Stevedore knot is slice but not double-slice (see [Su]). More generally, following [Su], we say that $K$ is invertibly concordant to $K'$ if there is a concordance $\mathcal{V}$ from $K$ to $K'$ and a concordance $\mathcal{W}$ from $K'$ to $K$ such that if we stack $\mathcal{W}$ on top of $\mathcal{V}$, the resulting concordance from $K$ to itself is diffeomorphic to the product concordance $(I \times S^2, I \times K)$. If we write $K \leq K'$, then $\leq$ is transitive and reflexive and perhaps even a partial ordering. It is easy to see that $0 \leq K$, where $0$ denotes the trivial knot, if and only if $K$ is double-slice.

Let $c^{1,\text{nf}}$ denote the subclass of $c^1$ consisting of clesses with no forks---a fork is a trivalent vertex two of whose incident edges contain a univalent vertex. Then, we will prove:

**Theorem 6.** If $G$ is a clover in the class $c^{1,\text{nf}}$ and $K'$ is obtained from $K$ by surgery on $G$ then $K \leq K'$.

It is natural to ask whether the converses to Theorems 3, 5 and 6 are true. If that were the case, one could extract from the rational functions invariants of [GK] many concordance invariants of knots. It was a bit of a surprise for us to show that the converses are all false.

First of all, it will follow easily from a recent result of Livingston that:

**Proposition 1.1.** There are $S$-equivalent knots which are not $c^2$-equivalent.

Then we will generalize some techniques of Kricker to prove:

**Theorem 7.** There are double-slice knots which are not $c^1$-equivalent to the unknot.

**Remark 1.2.** The proofs of Proposition 1.1 and Theorem 7 allow one to easily construct specific knots with the desired properties. See [Li, Theorem 10.1] for knots that satisfy Proposition 1.1. For the $(5, 2)$-torus knot $T_{5, 2}$, we have that $T_{5, 2} \# (-T_{5, 2})$ is a knot that satisfies Theorem 7.

1.3. Plan of the proof. Theorems 5 and 6 follow from an analysis of the surgery link corresponding to a clover.

Proposition 1.1 follows easily from the fact (proven recently by Livingston [Li], using Casson-Gordon invariants) that $S$-equivalence does not imply concordance.

Theorem 7 follows from the fact that under surgery on $c^1$-clesses, the Alexander polynomial changes under a more restrictive way than under a concordance.

2. Proofs

2.1. Proof of Theorem 5. Suppose that $G$ is a connected clover of class $c^1$ and $L$ is its associated framed link, [Gu, H, GGP]. We want to show that the knot $K'$ obtained from $K$ by surgery on $L$ is concordant to $K$. Note that the manifold $M$ obtained from $S^3$ by surgery on $L$ is diffeomorphic to $S^3$, see [Gu, H, GGP].

**Lemma 2.1.** We can express $L$ as a union of two sublinks $L'$ and $L''$ such that:

- $L'$ is a trivial 0-framed link in $S^3 - K$,
- $L''$ is a trivial 0-framed link in $S^3$.

Assuming this lemma we can complete the proof of Theorem 5 as follows.

Consider $I \times K \subset I \times S^3$ and $\frac{1}{2} \times L \subset \frac{1}{2} \times (S^3 - K)$. Consider a union of disjoint disks $D'$ in $\frac{1}{2} \times (S^3 - K)$ bounded by $L'$ and push their interiors into $[0, \frac{1}{2}) \times (S^3 - K)$. Also consider a union of disjoint disks $D''$ in $\frac{1}{2} \times S^3$ bounded by $L''$ and push their interiors into $(\frac{1}{2}, 1] \times S^3$. Now let $X \subset I \times S^3$ be obtained from $[0, \frac{1}{2}] \times S^3$ by removing a tubular neighborhood of $D'$ and adjoining a tubular neighborhood of $D''$. The boundary of $X$ consists of $0 \times S^3$ and a copy of $M$, which is diffeomorphic to $S^3$. Thus $X$ is diffeomorphic to $I \times S^3$ (indeed, add a $D^4$ to $X$ along $0 \times S^3$ and observe that any two imbeddings of a 4-disk in a fixed 4-disk.
are isotopic). Moreover $X$ contains $[0, \frac{1}{2}] \times K$, which is a concordance from $0 \times K \subset 0 \times S^3$ to $\frac{1}{2} \times K \subset M$, which is just $K'$.

**Proof of Lemma 2.1.** This is a generalization of the argument used to prove Theorem 2 in [Le2]. Recall (e.g., from [GGP, Section 2.3]) that surgery on a clover $G$ with $n$ edges corresponds to surgery on a link $L$ of $2n$ components. Given an orientation of the edges of $G$, we can split $L$ into the disjoint union of $n$-component sublinks $L'$ and $L''$, where $L'$ (resp., $L''$) consists of the sublink of $L$ assigned to the tails of the edges of $G$ (resp., of the heads of the edges of $G$, together with the leaves of $G$). As long as we avoid assigning all three of the components at a trivalent vertex to $L'$ or $L''$, we will have the desired decomposition of $L$. The corresponding conditions imposed on the orientation of the edges of $G$ are:

1. No trivalent vertex is a source or a sink,
2. Every edge with a univalent vertex is oriented toward the univalent vertex.

These are the same conditions as (i) and (ii) in the proof of Theorem 2 in [Le2] except that we now require no trivalent sinks also. But this will follow by the same argument as in [Le2] except that we need to choose the orientations of the cut edges more carefully. In particular we need to avoid choosing the orientation of two cut edges which share a trivalent vertex so that they both point into that vertex. But it is not hard to see that this can be done.

The next two remarks are an addendum to Theorem 5.

**Remark 2.2.** Observe that the sublinks $L'$ and $L''$ of $L$ which are constructed from $G$ have the same number of components, and that the linking matrix of $L$ is hyperbolic. Lemma 2.1 is analogous to the case of a knot which bounds a Seifert surface with a metabolic Seifert surface. In that case, the knot is algebraically slice, and if a metabolizer can be chosen to be bands of the Seifert surface that form a slice link, then the knot is slice.

**Remark 2.3.** Suppose that a knot $K'$ is obtained from the unknot $K$ by surgery on a connected clover of class $c^1$. It follows from Theorem 5 that $K'$ is slice. Using the calculus of clovers, one can show that $K'$ is actually ribbon, as observed also by Kricker and Habiro.

2.2. **Proof of Theorem 6.** We need a refinement of Lemma 2.1. Consider a connected clover $G$ of class $c^{1,nt}$, and let $L$ be its associated framed link.

**Lemma 2.4.** There is a link $\tilde{L}$ in $S^3 - K$, Kirby equivalent to $L$ in $S^3 - K$, so that $\tilde{L}$ is a union of two sublinks $\tilde{L}'$, $\tilde{L}''$, each of which is trivial in $S^3 - K$.

Assuming this lemma, we finish the proof following the lines of the argument following Lemma 2.1. The only difference is that we now use $\tilde{L}$ instead of $L$ and that $X' = \tilde{L} \times S^3 - \overline{X}$, which is also diffeomorphic to $\tilde{L} \times S^3$, now also contains $[\frac{1}{2}, 1] \times K$. Thus $M$ splits the trivial concordance from $K$ to itself. This, by definition, means $K \preceq K'$.

**Proof of Lemma 2.4.** For each univalent vertex of $G$, there is a corresponding part of $L$ which looks like the left part of Figure 1.

![Figure 1](image)

**Figure 1.** The associated link of a clover near a univalent vertex which is not a fork, before and after a Kirby move.

Now we can perform a Kirby move (see [Kr], [MO]) so that the four component link $\{L_1, \ldots, L_4\}$ in Figure 1 is replaced by two component link $\{\overline{L}_3, \overline{L}_4\}$. If we do this at every univalent vertex of $G$ we obtain the
link \( \tilde{L} \). Now consider the partition \( L = L' \cup L'' \) given by Lemma 2.1. The corresponding partition of \( \overline{L} \) is given by \( \overline{L}' = \{ \overline{K} | K \in L' - \{L_1, L_2\} \} \) and \( \overline{L}'' = \{ \overline{K} | K \in L'' - \{L_1, L_2\} \} \). It is easy to see that both \( \overline{L}' \) and \( \overline{L}'' \) are trivial in \( S^3 - K \). This completes the proof. \( \square \)

2.3. Proof of Proposition 1.1. Assume that \( S \)-equivalence implies \( c^2 \) on \( K \). Since \( c^2 \) implies \( c^1 \), and \( c^1 \) implies concordance (by Theorem 3), it follows that \( S \)-equivalence implies concordance. This is false. Livingston using Casson-Gordon invariants, shows that there are \( S \)-equivalent knots which are algebraically slice, but not slice, [Li, Theorem 0.4]. Since Livingston uses Casson-Gordon invariants, his examples have nontrivial Alexander module.

2.4. Proof of Theorem 7. We show that the Alexander polynomial \( \Delta \) of a knot changes in a more restrictive way under \( c^1 \)-equivalence than under concordance. Recall that if \( K \) and \( K' \) are concordant knots, then their Alexander polynomials satisfy \( \Delta_K(t) \theta(t) \theta^{-1} = \Delta_K(t) \theta(t) \theta^{-1} \) for some \( \theta(t), \theta(t) \in \mathbb{Z}[t, t^{-1}] \) satisfying \( \theta(1) = \theta'(1) = \pm 1 \). Moreover, there are double-slice knots with Alexander polynomial \( \theta(t) \theta^{-1} \) for any such \( \theta \). On the other hand,

**Lemma 2.5.** Let \( K \) and \( K' \) be \( c^1 \)-equivalent knots. Then,

\[
\Delta_K(t) \theta(t) \theta^{-1} = \Delta_K(t) \theta(t) \theta^{-1}
\]

where \( \theta(t) \) and \( \theta(t) \) are products of polynomials of the form \( 1 \pm t^k(t-1)^n \) for some integers \( k, n \) with \( n > 0 \).

**Proof.** We prove this using a generalization of an argument of Kricker [Kr]. Consider a connected clover \( G \) of the class \( c^1 \). Suppose that \( K \) is obtained from \( K \) by surgery on \( G \). If \( G \) has at least one internal trivalent vertex, then \( K \) and \( K' \) are \( S \)-equivalent (see the discussion following Theorem 3); in particular \( \Delta_K(t) = \Delta_K(t) \). Otherwise, \( G \) must be a wheel with a certain number \( n \) of legs and with a total of \( 2n \) edges. Thus, the associated link \( L' \) in \( S^3 - K \) has 4n components (see Figure below). Using the Kirby move in Figure 1 at every leaf of \( G \) we see that \( L' \) is Kirby-equivalent to a link \( L \) with 2n components, whose components can be numbered in pairs \( l_1, r_1, \ldots, l_n, r_n \) so that:

1. \( l_i \) (resp. \( r_i \)) bounds a disk \( d_i \) (resp. \( e_i \)) in \( S^3 - K \);
2. \( d_i \cap e_i \), for \( 1 \leq i \leq n \), each consists of two oppositely oriented clasps,
3. \( e_i \cap d_{i+1} \), for \( 1 \leq i < n \) and \( e_n \cap d_1 \) each consists of a single clasp, and
4. there are no other intersections among the disks.

An example for \( n = 2 \) is shown below:

![Example diagram](image)

We can now lift \( d_i \) and \( e_i \) to disks, \( \tilde{d}_i \) and \( \tilde{e}_i \), in the infinite cyclic cover \( \tilde{X} \) of \( X = S^3 - K \). The lifts of \( l_i, r_i \) form a link \( \tilde{L} \) in \( \tilde{X} \) which has a linking matrix \( B \) with entries in \( \mathbb{Z}[t, t^{-1}] \). To compute \( B \) note that we can choose the lifts \( \tilde{d}_i \) and \( \tilde{e}_i \) so that:

1. \( \tilde{d}_i \cap \tilde{e}_i \) consists of a single clasp, for every \( i \),
2. \( \tilde{d}_i \cap t \tilde{e}_i \) consists of a single clasp, oriented opposite to that in (1), for every \( i \),
3. \( \tilde{e}_i \cap \tilde{d}_{i+1} \), for \( 1 \leq i < n \), consists of a single clasp, and
4. \( \tilde{e}_n \cap t^k(\tilde{d}_1) \), for some integer \( k \), consists of a single clasp.

In (4), \( k \) (up to sign) is just the linking number of \( K \) with the imbedded wheel of \( G \).
Now it follows from this intersection data and the fact that $L$ is 0-framed that we can orient $L$ so that the linking matrix $B$ is given by

$$B = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix} \quad \text{where} \quad D = \begin{pmatrix} t - 1 & 1 & 0 & \cdots & 0 \\ 0 & t - 1 & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t - 1 & 1 \\ \pm t^k & 0 & \cdots & 0 & t - 1 \end{pmatrix}.$$ 

For any matrix $A$ over $\mathbb{Z}[t, t^{-1}]$, $A^*$ denotes the conjugate (under the involution $t \leftrightarrow t^{-1}$) transpose of $A$. The desired result $\Delta_{K'}(t) = \Delta_K(t)\theta(t)\theta(t^{-1})$ is now a consequence of the following lemma, which is proved by a standard argument going back to Kervaire-Milnor, generalized to covering spaces (see for example [Le1, p.140]).

Suppose $K \subset S^3$ is a knot, $L$ a framed link in $X = S^3 - K$, and $K' \subset S^3_L$ the knot produced from $K$ by surgery on $L$. Assume that the components of $L$ are null-homologous in $X$ and the components of $\tilde{L} \subset \tilde{X}$, the lift of $L$ into $\tilde{X}$, are null-homologous. In this case we have well-defined linking numbers of the components of $\tilde{L}$ which are organized into a matrix $B$ with entries in $\mathbb{Z}[t, t^{-1}]$ in the usual way. Let $A(K) = H_1(\tilde{X})$ and $A(K') = H_1(\tilde{Y})$ denote the Alexander modules of $K, K'$, where $Y = S^3_L - K'$.

**Lemma 2.6.** There is an exact sequence of $\mathbb{Z}[t, t^{-1}]$-modules

$$0 \to M \to A(K') \to A(K) \to 0$$

where $M$ is a module with presentation matrix $B$. In particular, $\Delta_{K'} = \Delta_K \det(B)$.

**Proof.** Observe that $\tilde{Y} = \tilde{X}_L$. Consider the following diagram of exact sequences of $\mathbb{Z}[t, t^{-1}]$-modules.

$$\begin{array}{ccccccccc}
H_2(\tilde{Y}, \tilde{X} - \tilde{L}) & \to & H_2(\tilde{X}, \tilde{X} - \tilde{L}) & \to & H_1(\tilde{X} - \tilde{L}) & \to & H_1(\tilde{X}) & \to & H_1(\tilde{X}, \tilde{X} - \tilde{L}) \\
\downarrow \psi & & \downarrow & & \downarrow i_* & & \downarrow & & \\
H_1(\tilde{Y}) & \to & H_1(\tilde{Y}, \tilde{X} - \tilde{L}) \\
\downarrow & & & & & & & & \\
H_1(\tilde{Y}, \tilde{X} - \tilde{L})
\end{array}$$

Notice that $H_1(\tilde{X}, \tilde{X} - \tilde{L}) = H_1(\tilde{Y}, \tilde{X} - \tilde{L}) = 0$. Moreover, $H_2(\tilde{X}, \tilde{X} - \tilde{L})$ is freely generated by the meridian disks of $L$, lifted to $\tilde{X}$, and $H_2(\tilde{Y}, \tilde{X} - \tilde{L})$ is freely generated by the disks attached by the surgeries. Thus, since the components of $\tilde{L}$ are null-homologous in $\tilde{X}$, $i_* \circ \partial_* = 0$. Also note that $H_2(\tilde{X}) = 0$ and so we have a mapping

$$H_2(\tilde{Y}, \tilde{X} - \tilde{L}) \to H_2(\tilde{X}, \tilde{X} - \tilde{L})$$

induced by $\partial_*$, which can be interpreted as expressing the longitudes of $\tilde{L}$ as linear combinations of the meridians of $\tilde{L}$ in $H_1(\tilde{X} - \tilde{L})$. Therefore this map is given by the linking numbers of $\tilde{L}$ and has $B$ as a representative matrix. This completes the proof of Lemma 2.6 and, as a consequence, Lemma 2.5. \hfill \Box

To complete the proof of Theorem 7 we need the following lemma.

**Lemma 2.7.** Let $f(t)$ be a polynomial of the form $1 \pm t^k(t - 1)^n$, for any integers $k, n$ with $n \neq 0$. Then any root of $f(t)$ which lies on the unit circle must be of the form $e^{\pm \pi i/3}$.

**Proof.** If $z$ is a root of $f(t)$ then $|z|^k|z - 1|^n = 1$. Thus we have $|z| = |z - 1| = 1$, from which the conclusion follows. \hfill \Box
Now choose some $\theta(t)$ with a root on the unit circle different from $e^{\pm \pi i/3}$ but with $\theta(1) = 1$—for example any cyclotomic polynomial of composite order not equal to 6. Let $K$ be a double-slice knot with Alexander polynomial $\theta(t)\theta(t^{-1})$ (see [Su, Theorem 3.3]). Then it follows from Lemmas 2.5 and 2.7 that $K$ is not equivalent to the trivial knot.

We end with a remark concerning the inverse of surgery on a wheel.

**Remark 2.8.** Recall that if a knot $K'$ is obtained from a knot $K$ by surgery on a Y-graph $G$, then there exists a Y-graph $G'$ such that $K$ is obtained from $K'$ by surgery on $G'$, see [GGP, Theorem 3.2]. Recall also that surgery on a wheel is described in terms of surgery on a union of Y-graphs, as explained in [GGP, Section 2.3]; in particular the inverse of surgery on a wheel can be described in terms of surgery on a union of Y-graphs. One might guess that the inverse of surgery on a wheel can be described in terms of surgery on a wheel. This is false, since the proof of Lemma 2.5 implies that if $K'$ is obtained from $K$ by surgery on a wheel $G$, then $\Delta_K$ always divides (and it can happen that it is not equal to) $\Delta_{K'}$.

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