

MULTIVARIABLE KNOT POLYNOMIALS FROM BRAIDED HOPF ALGEBRAS WITH AUTOMORPHISMS

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ABSTRACT. We construct knot invariants from solutions to the Yang–Baxter equation associated to appropriately generalized left/right Yetter–Drinfel’d modules over a braided Hopf algebra with an automorphism. When applied to Nichols algebras, our method reproduces known knot polynomials and naturally produces multivariable polynomial invariants of knots. We discuss in detail Nichols algebras of rank 1 and an example of rank 2. In the latter case, we compute the associated invariants for selected knots and pose some questions about their structure.

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1. INTRODUCTION

Jones's discovery of his famous polynomial of knots had an enormous influence in knot theory and connected the subject of low dimensional topology and hyperbolic geometry to mathematical physics, giving rise to quantum topology [Jon87, Thu77, Wit89].

The Jones polynomial was originally defined by thinking of a knot as the closure of a braid, and by taking the (suitably normalized) trace of representations of the braid groups (with an arbitrary number of strands), which themselves were determined by a vector space V and an automorphism $R \in \text{Aut}(V \otimes V)$ that satisfies the Yang–Baxter equation

$$R_1 R_2 R_1 = R_2 R_1 R_2 \in \text{End}(V \otimes V \otimes V), \quad (1)$$

where $R_1 = R \otimes I$, $R_2 = I \otimes R$.

It was soon realized that representations of simple Lie algebras and their deformations, known as quantum groups, were a natural source of solutions to the Yang–Baxter equations. This led to a plethora of polynomial invariants of knots; see for example Turaev [Tur88, RT90].

Another source of polynomial invariants (one for every complex root of unity) came from the work of Akutsu–Deguchi–Ohtsuki [ADO92]. It was conjectured by Habiro [Hab08, Conj.7.4] and later shown by Willets in [Wil22] that the collection of the colored Jones polynomials of a knot (colored by the irreducible representations of \mathfrak{sl}_2) determines and is determined by the collection of ADO invariants at roots of unity.

The definition of the above invariants requires an R -matrix together with a (ribbon) enhancement of it which, roughly speaking, is an endomorphism of V required to define the quantum trace, and hence the knot invariant. This comes from the Reshetikhin–Turaev functor which forms the basis of knot/link invariants in arbitrary 3-manifolds [RT90].

An R -matrix alone is in principle sufficient to define knot invariants. This was clarified by the second author by constructing invariants of knots from an R -matrix that satisfies some non-degeneracy conditions, called rigidity in [Kas21]. Rigid R -matrices indeed allow one to define state-sum invariants of planar projections of knots without any extra data. A description of how these invariants are defined is given in Section 2 below.

One can construct rigid R -matrices from any Hopf algebra with invertible antipode through Drinfel'd's quantum double construction which can be put into pure algebraic setting of multilinear algebra without finiteness assumptions of the Hopf algebra [Kas23].

In this paper, we propose a different approach of producing rigid R -matrices that does not use the quantum double of a Hopf algebra. The construction of these R -matrices, and the corresponding knot invariants, is schematically summarized in the following steps:

$$\left\{ \begin{array}{c} \text{Braided} \\ \text{Hopf algebras} \\ \text{with autos} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Braided} \\ \text{left/right YD modules} \\ \text{with automorphisms} \end{array} \right\} \rightarrow \{R\text{-matrices}\} \rightarrow \{\text{Knot invariants}\} \quad (2)$$

The last arrow in (2) is the well-known Reshetikhin–Turaev functor reviewed in Section 2. The first and the second arrows are discussed in Sections 3.1 and 3.2 below. These sections

are written in the maximum level of abstraction, using the language of category theory, for potential future applications to braided categories not coming from vector spaces.

The knot invariants defined by the steps in (2) require as input a braided Hopf algebra with automorphisms. A concrete source of braided Hopf algebras with a rich group of automorphisms are the Nichols algebras discussed in detail in Section 5. Roughly speaking, a Nichols algebra is the quotient of a naturally graded tensor algebra of a braided vector space by a suitable grading preserving maximal Hopf ideal. Any choice of an automorphism gives rise to appropriately generalized left and right Yetter–Drinfel’d module structures over the Nichols algebra. It turns out that the underlying Nichols algebra as a braided vector space admits natural quotient space in the case of the left generalized Yetter–Drinfel’d module and a natural submodule in the case of the right generalized Yetter–Drinfel’d module. In a sense, the choice of the braided Hopf algebra automorphism seems to correspond to the choice of a quantum double representation in the traditional approach.

In Sections 6 and 7 we study these generalized Yetter–Drinfel’d modules of Nichols algebras of diagonal type in the cases when the input braided vector space is of dimension one and two, which gives rise respectively to one and two-variable polynomial invariants of knots.

We end this introduction with some further comments.

1. An important feature of our construction is a braided Hopf algebra with an automorphism. The nontriviality of the automorphism is an essential part for constructing nontrivial knot invariants. In a sense the group of automorphisms replaces the representation theory.
2. Our approach unifies previous constructions of knot invariants (notably the colored Jones and the ADO polynomials, the Links–Gould and the Harper polynomials) coming from super/quantum groups, but also leads to a systematic construction of multivariable polynomial invariants of knots beyond the quantum groups.
3. A feature of the knot polynomials that we construct is that they depend on variables coming both from the braiding and the automorphism of the braided Hopf algebra. It is likely that some of our polynomial invariants of knots coincide with conjectured knot invariants that are discussed in the physics literature; see for instance the work of Gukov et al [GHN⁺21].
4. We expect that some of these invariants come from finite type invariants of knots [BN95], though we have not investigated this at the moment.
5. We expect that our polynomial invariants give lower bounds for the Seifert genus of a knot (as is known for the classical Alexander polynomial, but also in some other cases already, see [NvdV, KT]).
6. Regarding q -holonomic aspects, we expect our invariants to satisfy the analogous q -holonomic properties (that is, linear q -difference equations), as those that come from quantum groups (such as the colored Jones polynomials associated to a simple Lie algebra and parametrized by weights of irreducible representations [GL05]) or those defined at roots of unity (such as the ADO invariant [BDGG]).
7. Finally, regarding asymptotic aspects, we expect that our invariants satisfy versions of the Volume Conjecture, analogous to those of the colored Jones polynomials [Kas97, MM01] and the ADO invariants [Mur08].

2. FROM R -MATRICES TO KNOT INVARIANTS

In this section we briefly describe the Reshetikhin–Turaev functor which allows to construct knot invariants from R -matrices. These invariants are defined by state-sums [RT90], using a variation of the construction from the second author’s paper [Kas21]. There are three ingredients involved in this construction, namely suitable knot diagrams, rigid R -matrices, and the corresponding state-sums.

2.1. Knot diagrams and rigid R -matrices. We use a diagrammatic notation which is very important for the construction of knot invariants and has a long and successful history in knot theory [RT90, Tur94]. Basically, knots are represented by generic planar projections composed of local pieces which correspond to structural morphisms of a braided vector space, while the compatibility conditions ensure invariance under changes of planar projections. The notation leads naturally to the concept of a braided monoidal category, not necessarily in an abelian category, that vastly generalizes the notion of a braided vector space [TV17].

Following [Kas21], we now explain concretely the knot diagrams used. An (oriented) **long knot diagram** K is an oriented knot diagram in \mathbb{R}^2 with two open ends called “in” and “out”:

$$K = \begin{array}{c} \uparrow \text{out} \\ \boxed{K} \\ \downarrow \text{in} \end{array} \quad \text{Examples:} \quad K = \text{figure-eight}, \quad K = \text{trefoil}.$$

A long knot diagram can be closed to a planar projection of a knot: $\begin{array}{c} \uparrow \\ \boxed{K} \\ \downarrow \end{array} \mapsto \text{closed knot}.$

The vertical direction plays a preferred role for long knot diagrams.

The **normalization** \dot{K} of K is the diagram obtained from K by the replacements of local extrema oriented from left to right

$$\text{local extrema} \mapsto \text{normalized diagrams} \quad \text{and} \quad \text{local extrema} \mapsto \text{normalized diagrams} \quad (3)$$

at all possible locations of K . We say that K is normal if $K = \dot{K}$.

The **building blocks** of normal diagrams are given by four types of segments

$$\uparrow, \downarrow, \text{curved up}, \text{curved down} \quad (4)$$

and eight types of crossings (four positive and four negative ones)

$$\text{crossings} \quad (5)$$

We next define R -matrices and their rigid version. An **R -matrix** over a vector space V is an automorphism $r \in \text{Aut}(V \otimes V)$ of $V \otimes V$ that satisfies the quantum **Yang–Baxter relation**

$$r_1 r_2 r_1 = r_2 r_1 r_2, \quad r_1 := r \otimes \text{id}_V, \quad r_2 := \text{id}_V \otimes r. \quad (6)$$

Let V^* denote the dual vector space and $\langle \cdot, \cdot \rangle : V^* \otimes V \rightarrow \mathbb{F}$ denote the natural evaluation map. Assume that V is a finite-dimensional, and fix a basis B of V and the corresponding dual basis $\{b^*\}_{b \in B}$ of V^* .

assume the preference of the composition over the monoidal product, so that, for example, $fg \otimes h$ will mean $(fg) \otimes h$.

When a functor $F: \mathcal{D} \rightarrow \mathcal{C}$ is considered as an object of the functorial category $\mathcal{C}^{\mathcal{D}}$, it will be called *functorial object* or just *f-object* for brevity.

Let \mathcal{C} be a braided monoidal category. Denote by $\mathcal{C}^{\mathbb{Z}}$ the braided monoidal category of functors $F: \mathbb{Z} \rightarrow \mathcal{C}$ where the additive group of integers \mathbb{Z} is viewed as a category with one object $*$ whose automorphism group is \mathbb{Z} . We denote by $\tau: \otimes \rightarrow \otimes^{\text{op}}$ the braiding of $\mathcal{C}^{\mathbb{Z}}$ which assigns to any pair of f-objects F and G a functorial morphism $\tau_{F,G}: F \otimes G \rightarrow G \otimes F$ that at the unique object $*$ of \mathbb{Z} evaluates to the morphism of \mathcal{C}

$$(\tau_{\mathcal{C}})_{F(*),G(*)}: F(*) \otimes G(*) \rightarrow G(*) \otimes F(*)$$

where $\tau_{\mathcal{C}}$ is the braiding in \mathcal{C} .

Remark 3.1. Given the fact that the group \mathbb{Z} is freely generated by one element 1, an object G of the functor category $\mathcal{C}^{\mathbb{Z}}$ is uniquely determined by the pair (A, ϕ) where A is the object of \mathcal{C} obtained as the image by G of the unique object $*$ of \mathbb{Z} , and $\phi: A \rightarrow A$ is the automorphism of A obtained as the image by G of the generating element 1 of \mathbb{Z} . With this interpretation, a morphism from (A, ϕ) to (B, ψ) is a morphism $f: A \rightarrow B$ in \mathcal{C} such that $\psi f = f \phi$. The monoidal product of two pairs $(A, \phi) \otimes (B, \psi)$ is given by the pair $(A \otimes B, \phi \otimes \psi)$.

Definition 3.2. A *Hopf f-object* is an f-object $H: \mathbb{Z} \rightarrow \mathcal{C}$ together with functorial morphisms (natural transformations)

$$\nabla: H \otimes H \rightarrow H, \quad \eta: \mathbb{I} \rightarrow H, \quad \Delta: H \rightarrow H \otimes H, \quad \epsilon: H \rightarrow \mathbb{I}, \quad S: H \rightarrow H \quad (11)$$

such that (H, ∇, η) is an algebra f-object, (H, Δ, ϵ) is a coalgebra f-object and

$$(\nabla \otimes \nabla)(\text{id}_H \otimes \tau_{H,H} \otimes \text{id}_H)(\Delta \otimes \Delta) = \Delta \nabla, \quad (12)$$

$$\nabla(S \otimes \text{id}_H)\Delta = \eta\epsilon = \nabla(\text{id}_H \otimes S)\Delta. \quad (13)$$

We will always assume that S is an invertible (functorial) morphism. As in the theory of Hopf algebras, the functorial morphisms $\nabla, \eta, \Delta, \epsilon$ and S are respectively called product, unit, coproduct, counit and antipode.

Definition 3.3. Let $H: \mathbb{Z} \rightarrow \mathcal{C}$ be a Hopf f-object. A *left Yetter–Drinfel’d f-object* over H is a triple (Y, λ, δ) where $Y: \mathbb{Z} \rightarrow \mathcal{C}$ is an f-object of $\mathcal{C}^{\mathbb{Z}}$, and $\lambda: H \otimes Y \rightarrow Y, \delta: Y \rightarrow H \otimes Y$ are morphisms of $\mathcal{C}^{\mathbb{Z}}$ such that (Y, λ) is a left H -module f-object, (Y, δ) is a left H -comodule f-object, and

$$\begin{aligned} (\nabla \otimes \text{id}_Y)(\text{id}_H \otimes \tau_{Y,H})(\delta \lambda \otimes \phi_H)(\text{id}_H \otimes \tau_{H,Y})(\Delta \otimes \text{id}_Y) \\ = (\nabla \otimes \lambda)(\text{id}_H \otimes \tau_{H,H} \otimes \text{id}_Y)(\Delta \otimes \delta) \end{aligned} \quad (14)$$

where $\phi_H: H \rightarrow H$ is the functorial isomorphism that at the unique object $*$ of \mathbb{Z} evaluates as

$$(\phi_H)_* = H(1): H(*) \rightarrow H(*)$$

Taking into account the self-dual nature of Hopf objects, it is useful to have the dual version of Definition 3.3 which reads as follows.

Definition 3.4. A *right Yetter–Drinfel’d f-object* over a Hopf f-object $H: \mathbb{Z} \rightarrow \mathcal{C}$ is a triple (Y, λ, δ) where $Y: \mathbb{Z} \rightarrow \mathcal{C}$ is an f-object of $\mathcal{C}^{\mathbb{Z}}$, and $\lambda: Y \otimes H \rightarrow Y$, $\delta: Y \rightarrow Y \otimes H$ are functorial morphisms of $\mathcal{C}^{\mathbb{Z}}$ such that (Y, λ) is a right H -module f-object, (Y, δ) is a right H -comodule f-object, and

$$\begin{aligned} & (\text{id}_Y \otimes \nabla)(\tau_{H,Y} \otimes \text{id}_H)(\phi_H \otimes \delta\lambda)(\tau_{Y,H} \otimes \text{id}_H)(\text{id}_Y \otimes \Delta) \\ & = (\lambda \otimes \nabla)(\text{id}_Y \otimes \tau_{H,H} \otimes \text{id}_H)(\delta \otimes \Delta). \end{aligned} \quad (15)$$

We will return and give further clarifications to these definitions later in Subsection 4.2 after introducing the graphical notation of string diagrams.

For a Hopf f-object $H: \mathbb{Z} \rightarrow \mathcal{C}$, we denote by $\Delta^{(2)}$ and $\nabla^{(2)}$ the twice iterated coproduct and product, respectively, defined by

$$\begin{aligned} \nabla^{(2)}: H \otimes H \otimes H &\rightarrow H, & \nabla^{(2)} &= \nabla(\nabla \otimes \text{id}_H) \\ \Delta^{(2)}: H &\rightarrow H \otimes H \otimes H, & \Delta^{(2)} &= (\Delta \otimes \text{id}_H)\Delta. \end{aligned} \quad (16)$$

The following theorem provides constructions of left/right Yetter–Drinfel’d f-objects over a Hopf f-object $H: \mathbb{Z} \rightarrow \mathcal{C}$.

Theorem 3.5. *For any Hopf f-object $H: \mathbb{Z} \rightarrow \mathcal{C}$,*

(a) *the triple (H, ∇, δ) is a left Yetter–Drinfel’d f-object over H , where*

$$\delta := (\nabla \otimes \text{id}_H)(\text{id}_H \otimes \tau_{H,H})(\text{id}_{H \otimes H} \otimes S\phi_H)\Delta^{(2)}; \quad (17)$$

(b) *the triple (H, λ, Δ) is a right Yetter–Drinfel’d f-object over H , where*

$$\lambda := \nabla^{(2)}(S\phi_H \otimes \text{id}_{H \otimes H})(\tau_{H,H} \otimes \text{id}_H)(\text{id}_H \otimes \Delta). \quad (18)$$

3.2. From left/right Yetter–Drinfel’d f-objects to R -matrices. The next theorem constructs R -matrices from left/right Yetter–Drinfel’d f-objects corresponding to the second arrow in (2).

Theorem 3.6. *Let $H: \mathbb{Z} \rightarrow \mathcal{C}$ be a Hopf f-object and (Y, λ, δ) be a left, respectively a right, Yetter–Drinfel’d f-object over H . Then*

$${}^L\rho = (\lambda \otimes \text{id}_Y)(\text{id}_H \otimes \tau_{Y,Y})(\delta \otimes \phi_Y), \quad (19)$$

respectively

$$\rho^R = (\phi_Y \otimes \lambda)(\tau_{Y,Y} \otimes \text{id}_H)(\text{id}_Y \otimes \delta), \quad (20)$$

is an R -matrix, that is a solution of the following braid group type Yang–Baxter relation in the automorphism group $\text{Aut}(Y \otimes Y \otimes Y)$:

$$(\rho \otimes \text{id}_Y)(\text{id}_Y \otimes \rho)(\rho \otimes \text{id}_Y) = (\text{id}_Y \otimes \rho)(\rho \otimes \text{id}_Y)(\text{id}_Y \otimes \rho). \quad (21)$$

Moreover, this R -matrix is rigid if the f-object Y is rigid.

The proof of these theorems is given in the next section, using a diagrammatic calculus.

A corollary of Theorem (3.6) gives an invariant of knots.

Theorem 3.7. *Fix a rigid left or a right Yetter–Drinfel’d f-object Y over a Hopf f-object H . Then, there exists a knot invariant*

$$\{\text{Knots in } S^3\} \rightarrow \text{End}(Y), \quad K \mapsto W_K^Y. \quad (22)$$

4. PROOFS

4.1. Diagrammatics of braided Hopf algebras with automorphisms. The Hopf objects introduced in Section 3 are categorical versions of pairs (H, ϕ) where H is a braided Hopf algebra and ϕ is an automorphism of H . At around the same time of the Reshetikhin–Turaev construction of knot invariants via diagrammatics, there was a parallel intense activity in the theory of Hopf algebras motivated in part by the theory of quantum groups developed by Drinfel’d and Jimbo [Jim86]. There is a string diagrammatic calculus designed to prove tensor identities in Hopf algebras that avoids using explicit coordinate formulas for the tensors involved.

This string diagrammatic calculus extends to the case of braided Hopf algebras, introduced by Majid around 1990 [Maj94, Maj95], and used extensively by many authors including Radford, Kuperberg and Kauffman [Rad12, Kup91, KR95]. A survey of the various directions of braided Hopf algebras around 2000 is given by Takeuchi in [Tak00].

The string diagrammatics of the generators and relations of a Hopf algebra are given in [Maj95]. For a recent treatment, see [Kas23], namely, Eqns. (1.81)–(1.85) for the generators, Eqns. (1.86)–(1.91) for the relations and Eqns. (1.68)–(1.73) for the diagrammatic notation. For the convenience of the reader, we recall below the definitions of these morphisms, relations, and the string diagrammatic notation.

Let \mathcal{C} be a category. To any morphism $f: X \rightarrow Y$ in \mathcal{C} , we associate a graphical picture

$$f =: \begin{array}{c} Y \\ | \\ \boxed{f} \\ | \\ X \end{array}. \quad (23)$$

If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are two composable morphisms, then their composition is described by the vertical concatenation of graphs

$$g \circ f = \begin{array}{c} Z \\ | \\ \boxed{g \circ f} \\ | \\ X \end{array} = \begin{array}{c} Z \\ | \\ \boxed{g} \\ | \\ Y \\ | \\ \boxed{f} \\ | \\ X \end{array}. \quad (24)$$

In particular, for the identity morphism id_X it is natural to use just a line

$$\text{id}_X = \begin{array}{c} X \\ | \\ \boxed{\text{id}_X} \\ | \\ X \end{array} = \begin{array}{c} X \\ | \\ | \\ | \\ X \end{array}. \quad (25)$$

The string diagrams are especially useful in the case when \mathcal{C} is a strict monoidal category, because the tensor (monoidal) product can be drawn by the horizontal juxtaposition. Namely, for two morphisms $f: X \rightarrow Y$ and $g: U \rightarrow V$, their tensor product

$f \otimes g: X \otimes U \rightarrow Y \otimes V$ is drawn as follows:

$$f \otimes g = \begin{array}{c} Y \otimes V \\ | \\ \boxed{f \otimes g} \\ | \\ X \otimes U \end{array} = \begin{array}{cc} Y & V \\ | & | \\ \boxed{f \otimes g} & \\ | & | \\ X & U \end{array} = \begin{array}{cc} Y & V \\ | & | \\ \boxed{f} & \boxed{g} \\ | & | \\ X & U \end{array}. \quad (26)$$

For example, the graphical equalities

$$\begin{array}{cc} Y & V \\ | & | \\ \boxed{f} & \boxed{g} \\ | & | \\ X & U \end{array} = \begin{array}{cc} Y & V \\ | & | \\ \boxed{f} & \\ | & | \\ X & U \end{array} \boxed{g} = \begin{array}{cc} Y & V \\ | & | \\ \boxed{f} & \\ | & | \\ X & U \end{array} \boxed{g} \quad (27)$$

correspond to the well known relations in the tensor calculus

$$f \otimes g = (f \otimes \text{id}_V)(\text{id}_X \otimes g) = (\text{id}_Y \otimes g)(f \otimes \text{id}_U). \quad (28)$$

By taking into account the distinguished role of the identity object I , it is natural to associate to it the empty graph.

Let \mathcal{C} be a symmetric monoidal category with tensor product \otimes , the opposite tensor product \otimes^{op} , unit object I and symmetry $\sigma: \otimes \rightarrow \otimes^{\text{op}}$. Recall that a Hopf object in \mathcal{C} is an object H endowed with five structural morphisms

$$\nabla: H \otimes H \rightarrow H, \quad \eta: I \rightarrow H, \quad \Delta: H \rightarrow H \otimes H, \quad \epsilon: H \rightarrow I, \quad S: H \rightarrow H \quad (29)$$

called, respectively, product, unit, coproduct, counit and antipode, that satisfy the following relations or axioms

$$\text{associativity : } \nabla(\nabla \otimes \text{id}_H) = \nabla(\text{id}_H \otimes \nabla) \quad (30a)$$

$$\text{coassociativity : } (\Delta \otimes \text{id}_H)\Delta = (\text{id}_H \otimes \Delta)\Delta \quad (30b)$$

$$\text{unitality : } \nabla(\eta \otimes \text{id}_H) = \text{id}_H = \nabla(\text{id}_H \otimes \eta) \quad (30c)$$

$$\text{counitality : } (\epsilon \otimes \text{id}_H)\Delta = \text{id}_H = (\text{id}_H \otimes \epsilon)\Delta \quad (30d)$$

$$\text{invertibility : } \nabla(\text{id}_H \otimes S)\Delta = \eta\epsilon = \nabla(S \otimes \text{id}_H)\Delta \quad (30e)$$

$$\text{compatibility : } (\nabla \otimes \nabla)(\text{id}_H \otimes \sigma_{H,H} \otimes \text{id}_H)(\Delta \otimes \Delta) = \Delta\nabla. \quad (30f)$$

Let us introduce the following graphical notation for the structural maps of H (all lines correspond to the object H)

$$\nabla = \begin{array}{c} | \\ \boxed{\nabla} \\ | \end{array} = \begin{array}{c} | \\ \diagdown \quad \diagup \\ | \end{array} \quad (\text{product}), \quad \Delta = \begin{array}{c} \diagup \quad \diagdown \\ \boxed{\Delta} \\ | \end{array} = \begin{array}{c} | \\ \diagup \quad \diagdown \end{array} \quad (\text{coproduct}), \quad (31)$$

$$\eta = \begin{array}{c} | \\ \boxed{\eta} \\ \circ \end{array} = \begin{array}{c} | \\ \circ \end{array} \quad (\text{unit}), \quad \epsilon = \begin{array}{c} \bullet \\ \boxed{\epsilon} \\ | \end{array} = \begin{array}{c} \bullet \\ | \end{array} \quad (\text{counit}), \quad (32)$$

$$S = \begin{array}{c} | \\ \boxed{S} \\ | \end{array} = \begin{array}{c} | \\ \square \\ | \end{array} \quad (\text{antipode}). \quad (33)$$

We complete this with the graphical notation for the symmetry

$$\sigma_{H,H} = \boxed{\sigma_{H,H}} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}. \quad (34)$$

The relations or axioms of a Hopf object take the following graphical form:

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad (\text{associativity}), \quad \begin{array}{c} \diagup \\ \circ \end{array} = \begin{array}{c} | \\ \circ \end{array} = \begin{array}{c} \diagdown \\ \circ \end{array} \quad (\text{unitality}), \quad (35)$$

$$\begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad (\text{coassociativity}), \quad \begin{array}{c} \bullet \\ \diagdown \end{array} = \begin{array}{c} | \\ \bullet \end{array} = \begin{array}{c} \diagup \\ \bullet \end{array} \quad (\text{counitality}), \quad (36)$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \circ \\ | \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad (\text{invertibility}), \quad (37)$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad (\text{compatibility}). \quad (38)$$

Our first refinement is the notion of a braided Hopf algebra in a braided monoidal category. It generalizes the notion of a Hopf object (defined in the context of a symmetric monoidal category) by replacing the symmetry $\sigma_{H,H}$ in the compatibility axiom (30f) by the braiding $\tau = \tau_{H,H} : H \otimes H \rightarrow H \otimes H$ that satisfies the Yang–Baxter equation

$$(\tau \otimes \text{id}_H)(\text{id}_H \otimes \tau)(\tau \otimes \text{id}_H) = (\text{id}_H \otimes \tau)(\tau \otimes \text{id}_H)(\text{id}_H \otimes \tau). \quad (39)$$

In other words, a braided Hopf object (in a braided monoidal category) is defined by the same set of structural maps (29) that satisfy relations (30a)–(30e), while in the compatibility relation (30f) the symmetry σ is replaced by the braiding τ

$$\text{compatibility} : (\nabla \otimes \nabla)(\text{id}_H \otimes \tau_{H,H} \otimes \text{id}_H)(\Delta \otimes \Delta) = \Delta \nabla. \quad (40)$$

One can show that in any braided Hopf algebra, the antipode satisfies the relations

$$S \nabla = \nabla \tau_{H,H}(S \otimes S), \quad \Delta S = (S \otimes S) \tau_{H,H} \Delta \quad (41)$$

which can be proven, for example, following the same line of reasoning as in Section 1.9 of [Kas23].

In the diagrammatic language, we denote the braiding morphism by

$$\boxed{\tau_{H,H}} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad (42)$$

so that the compatibility relation (40) takes the graphical form

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad (43)$$

and relations (41) become

$$(44)$$

The second refinement that we need is the notion of a Hopf f-object or Hopf f-algebra which corresponds to a pair (H, ϕ) composed of a braided Hopf algebra H and a braided Hopf algebra automorphism $\phi : H \rightarrow H$. In addition to the axioms of a braided Hopf algebra for H , the pair (H, ϕ) satisfies the extra compatibility conditions between ϕ and all the structural morphisms of H :

$$\nabla(\phi \otimes \phi) = \phi \nabla, \quad (\phi \otimes \phi)\Delta = \Delta \phi, \quad (45)$$

$$S\phi = \phi S, \quad \phi\eta = \eta, \quad \epsilon\phi = \epsilon. \quad (46)$$

In the diagrammatic notation, we denote the automorphism ϕ by

$$(47)$$

so that the additional compatibility relations (45) and (46) take the form

$$(48)$$

$$(49)$$

4.2. Diagrammatics of Yetter-Drinfel'd f-objects. In this section we recall the definitions for Yetter-Drinfel'd f-objects over Hopf f-objects and provide the diagrammatic notation for them.

The original Yetter-Drinfel'd modules were defined by Yetter [Yet90] and they are essentially modules over Drinfel'd's quantum double of a Hopf algebra (hence the name of Drinfel'd). In the early literature, they were also called crossed modules; see eg. [Rad12, p. 385]. A detailed definition of these modules, their properties in the setting of braided Hopf algebras is given in Takeuchi [Tak00].

A left Yetter-Drinfel'd f-object over a Hopf f-object H is a triple (Y, λ, δ) where $\lambda : H \otimes Y \rightarrow Y$ and $\delta : Y \rightarrow H \otimes Y$ satisfy the left module and left comodule equations

$$\text{left action : } \lambda(\text{id}_H \otimes \lambda) = \lambda(\nabla \otimes \text{id}_Y) \quad (50a)$$

$$\text{left action of unit : } \lambda(\eta \otimes \text{id}_Y) = \text{id}_Y \quad (50b)$$

$$\text{left coaction : } (\text{id}_H \otimes \delta)\delta = (\Delta \otimes \text{id}_Y)\delta \quad (50c)$$

$$\text{left coaction of counit : } (\epsilon \otimes \text{id}_Y)\delta = \text{id}_Y \quad (50d)$$

In the diagrammatic setting, we will color the left f-objects by the blue color and the right f-objects by the red color. Using this coloring scheme, the morphisms λ and δ of the left Yetter-Drinfel'd f-objects are drawn graphically as

$$\boxed{\lambda} = \text{blue trivalent vertex with one blue line down and two black lines up}, \quad \boxed{\delta} = \text{blue trivalent vertex with one blue line down and two black lines up}, \quad (51)$$

so that we obtain the graphical form of Equations (50a)–(50b)

$$\text{blue trivalent vertex with one blue line down and two black lines up} = \text{blue trivalent vertex with one blue line down and two black lines up}, \quad \text{blue trivalent vertex with one blue line down and two black lines up} = \text{blue trivalent vertex with one blue line down and two black lines up}, \quad (52)$$

of Equations (50c)–(50d)

$$\text{blue trivalent vertex with one blue line down and two black lines up} = \text{blue trivalent vertex with one blue line down and two black lines up}, \quad \text{blue trivalent vertex with one blue line down and two black lines up} = \text{blue trivalent vertex with one blue line down and two black lines up}, \quad (53)$$

and the compatibility relation (14)

$$\text{blue trivalent vertex with one blue line down and two black lines up} = \text{blue trivalent vertex with one blue line down and two black lines up}. \quad (54)$$

Likewise, a right Yetter-Drinfel'd f-object over a Hopf f-object H is a triple (Y, λ, δ) where $\lambda: Y \otimes H \rightarrow Y$ and $\delta: Y \rightarrow Y \otimes H$ satisfy the right module and right comodule equations

$$\text{right coaction} : (\delta \otimes \text{id}_H)\delta = (\text{id}_Y \otimes \Delta)\delta \quad (55a)$$

$$\text{right coaction of counit} : (\text{id}_Y \otimes \epsilon)\delta = \text{id}_Y \quad (55b)$$

$$\text{right action} : \lambda(\lambda \otimes \text{id}_H) = \lambda(\text{id}_Y \otimes \Delta) \quad (55c)$$

$$\text{right action of unit} : \lambda(\text{id}_Y \otimes \eta) = \text{id}_Y \quad (55d)$$

The corresponding maps λ and δ of the right Yetter-Drinfel'd f-objects are denoted by

$$\boxed{\lambda} = \text{red trivalent vertex with one red line down and two black lines up}, \quad \boxed{\delta} = \text{red trivalent vertex with one red line down and two black lines up}, \quad (56)$$

so that we have graphical form of Equations (55a) and (55b)

$$\text{red trivalent vertex with one red line down and two black lines up} = \text{red trivalent vertex with one red line down and two black lines up}, \quad \text{red trivalent vertex with one red line down and two black lines up} = \text{red trivalent vertex with one red line down and two black lines up}, \quad (57)$$

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array}, \quad \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array}, \quad (58)$$
$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} . \quad (59)$$

4.3. Proof of Theorem 3.5. In this section we prove Theorem 3.5 using the diagrammatic language that we have already described. Before doing so, we need the following diagrammatic notation for the double-iterated coproduct (16)

$$\text{Diagram with } \Delta^{(2)} \text{ box} = \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} . \quad (60)$$

Using the following graphical representation of the left coaction (17)

$$\begin{array}{c} \text{diagram with } \delta \end{array} = \begin{array}{c} \text{diagram with black dot} \end{array} = \begin{array}{c} \text{diagram with loop and vertices} \end{array}, \quad (61)$$

$$\begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \end{array} = \begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \end{array} \quad (62)$$

. (63)

Similarly, we can prove the compatibility property (14), see (54) for its graphical form, which in this case takes the form

$$(64)$$

Indeed, with a bit longer graphical calculation, we have

$$(65)$$

This completes the proof of part (a) of Theorem 3.5. The proof of part (b) is analogous, and is omitted. \square

4.4. Proof of Theorem 3.6. In this section we show that the R -matrix (19) satisfies the Yang–Baxter equation (21), and omit the analogous proofs that the R -matrix (20) also satisfies the Yang–Baxter equation.

To begin with, the diagrammatic notation for the R -matrices ${}^L\rho$ and ρ^R is given as follows

$$(66)$$

The proof of Theorem 3.6 is now given as follows:

where the last equality is the compatibility Equation (54). This completes the proof of Theorem 3.5 for left Yetter–Drinfel’d f-objects. The proof of the right Yetter–Drinfel’d f-objects is obtained by rotating the above diagrams by 180 degrees, followed by replacing the blue color by the red color. \square

5. BRAIDED TENSOR ALGEBRAS AND NICHOLS ALGEBRAS

5.1. Braided tensor algebras. In this section we specialize the abstract language of Hopf f-objects and the Yetter–Drinfel’d f-objects to the context of a braided category \mathcal{C} which, as a monoidal category, is a subcategory of the category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over a field \mathbb{F} with the monoidal structure given by the tensor product $\otimes_{\mathbb{F}}$. The objects of \mathcal{C} will be called *braided vector spaces*. In this case, a Hopf f-object and a Yetter–Drinfel’d f-object will be respectively called a *Hopf f-algebra* and *Yetter–Drinfel’d f-module*.

Recall that it follows from the definition that a Hopf f-algebra is a pair (H, ϕ) of a braided Hopf algebra and an automorphism ϕ of it. There is an elementary universal construction of such pairs (H, ϕ) that we now discuss.

Fix a braided vector space V of finite dimension n and a basis B of V . Then, the tensor algebra $T(V)$ has a unique structure of a braided Hopf algebra determined by declaring all elements of V to be primitive. We define the rank of $T(V)$ to be the dimension of V , and call the braided Hopf algebra $T(V)$ to be of diagonal type if the braiding on V with respect to the basis B is diagonal.

In this case, $T(V)$ is $\mathbb{Z}_{\geq 0}^n$ -graded and admits a rich Abelian group of braided Hopf algebra automorphisms. Namely, any map $t: B \rightarrow \mathbb{F}_{\neq 0}$ corresponds to a braided Hopf algebra automorphism ϕ_t of $T(V)$ uniquely determined by

$$\phi_t b = t_b b, \quad \forall b \in B, \quad (68)$$

where we denote by t_b the image $t(b)$. We call such automorphisms scaling automorphisms.

Summarizing, a finite dimensional vector space V with a diagonal braiding with respect to a basis B of V , together with a map $t: B \rightarrow \mathbb{F}_{\neq 0}$ determines a pair $(T(V), \phi_t)$ of a

braided Hopf algebra and an automorphism of it. Using Theorems 3.5–3.6, we obtain multi-parameter infinite-dimensional R -matrices over the vector space $T(V)$. Our interest is to find rigid R -matrices which correspond to finite-dimensional Yetter–Drinfel’d f -modules. We discuss this next.

5.2. Nichols algebras. It turns out that the braided tensor algebras $T(V)$ defined above have a canonical quotient called Nichols algebra which is a braided Hopf algebra. It can be finite or infinite dimensional.

Recall that a *Nichols algebra* over a braided vector space V is the quotient braided Hopf algebra $\mathfrak{B}(V) = T(V)/J$ of the tensor algebra $T(V)$ over the maximal (braided) Hopf algebra ideal J intersecting trivially the part $\mathbb{F} \oplus V \subset T(V)$. In the case when the braiding is of diagonal type, the scaling automorphism ϕ_t of $T(V)$ descends to an automorphism of the braided Hopf algebra $\mathfrak{B}(V) = T(V)/J$ leading thereby to a Hopf f -algebra which we will call Nichols f -algebra. Thus, finite dimensional Nichols f -algebras can be used as an input to the construction of multiparameter knot invariants in (2).

Examples of finite dimensional Nichols algebras are the nilpotent Borel parts of Lusztig’s small quantum groups. A detailed description of finite dimensional Nichols algebras can be found for example in [AS00, AS02]. Like quantum groups, Nichols algebras have PBW bases [Kha99] and the ones of diagonal braiding have been classified by Heckenberger [Hec06, Hec09] building on the work of Kharchenko [Kha99] and Andruskiewitsch–Schneider [AS00]. The list of diagonal Nichols algebras of rank (that is, dimension of V) at most 3 is given in Tables 1 and 2 of [Hec06], and from this, it follows that the majority of finite rank Nichols algebras do not come from quantum groups. A presentation of Nichols algebras of diagonal type in terms of generators and relations is given by Angiono [Ang15].

5.3. Sub/quotient Yetter–Drinfel’d f -modules of Nichols f -algebras. If a Nichols algebra is infinite dimensional, we cannot immediately proceed to the construction of knot invariants.

It turns out that a Nichols f -algebra $\mathfrak{B}(V)$ has a canonical quotient ${}^L\mathfrak{B}(V)$ and a canonical subspace $\mathfrak{B}(V)^R$ which are left and right Yetter–Drinfel’d f -modules over $\mathfrak{B}(V)$ respectively. The construction of these f -modules is as follows:

$${}^L\mathfrak{B}(V) = \mathfrak{B}(V)/\mathfrak{B}(V)W_\delta, \quad W_\delta = \{x \in \mathfrak{B}(V) \setminus \mathbb{F} \mid \delta x = 1 \otimes x\} \quad (69)$$

where elements of W_δ are nothing else but the *coinvariant* elements in degree ≥ 1 with respect to the left coaction δ , see [Rad12, Def. 8.2.1]; and

$$\mathfrak{B}(V)^R = U_\lambda \quad (70)$$

where $U_\lambda \subset \mathfrak{B}(V)$ is the smallest subspace of $\mathfrak{B}(V)$ that satisfies

$$\Delta W_\lambda \subset U_\lambda \otimes \mathfrak{B}(V), \quad W_\lambda = \{x \in \mathfrak{B}(V) \mid \lambda(x \otimes y) = 0, \text{ for all } y \in \mathfrak{B}(V)\} \quad (71)$$

where, by taking into account the fact that $\epsilon x = 0$ for all $x \in V$, elements of W_λ are nothing else but *invariant* elements with respect to the right action λ , see [Rad12, Def. 11.2.3].

If any one of the $\mathfrak{B}(V)$ f -modules ${}^L\mathfrak{B}(V)$ or $\mathfrak{B}(V)^R$ is finite dimensional, then it can be used in (2) to construct polynomial knot invariants.

In the next sections we illustrate the quotients and the subspaces of a braided tensor algebra $T(V)$ when the dimension of V is 1 or 2.

6. THE RANK 1 TENSOR ALGEBRA

6.1. Definition. In this section we compute from first principles the R -matrices of Theorem 3.5 for the rank 1 tensor algebra, with no reference to Lie theory. As we will find out, the corresponding knot invariants are none other than the colored Jones and the ADO polynomials.

The rank 1 tensor algebra $T(\mathbb{F})$ is identified with the polynomial algebra $\mathbb{F}[x]$ in one indeterminate. It is an infinite dimensional \mathbb{F} -vector space with basis $B = \{x^k \mid k \in \mathbb{Z}_{\geq 0}\}$.

The Hopf algebra structure and the braided structure of $T(\mathbb{F})$ are determined by

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \tau(x \otimes x) = q x \otimes x. \quad (72)$$

The above equation, together with the axioms of a braided Hopf algebra, and the choice of the basis, uniquely determines the braided Hopf algebra structure. The formulas involve the q -Pochhammer symbol $(x; q)_n$ and the q -binomial coefficients $\begin{bmatrix} k \\ m \end{bmatrix}_q$ defined by

$$(x; q)_n := \prod_{i=0}^{n-1} (1 - xq^i), \quad \begin{bmatrix} k \\ m \end{bmatrix}_q := \frac{(q; q)_k}{(q; q)_{k-m} (q; q)_m}. \quad (73)$$

Explicitly, we have the following.

Lemma 6.1. The coproduct, the antipode and the scaling automorphism ϕ_t of $T(\mathbb{F})$ are given by

$$\Delta x^k = \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes x^m \quad (74)$$

$$Sx^k = (-1)^k q^{k(k-1)/2} x^k \quad (75)$$

$$\phi_t x^k = t^k x^k \quad (76)$$

respectively.

Proof. The primitivity of x implies that

$$\Delta x = x_1 + x_2, \quad x_1 := x \otimes 1, \quad x_2 := 1 \otimes x, \quad (77)$$

and the braiding implies that

$$x_2 x_1 = q x_1 x_2. \quad (78)$$

This, combined with the q -binomial formula, gives

$$\Delta x^k = (x_1 + x_2)^k = \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x_1^{k-m} x_2^m = \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes x^m. \quad (79)$$

This proves (74). To prove (75), apply (41) for $x^k \in T(\mathbb{F})$, use $\eta \epsilon x^k = \delta_{k,0}$ and compute

$$\begin{aligned} \nabla(\text{id}_H \otimes S) \Delta x^k &= \nabla(\text{id}_H \otimes S) \left(\sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes x^m \right) \\ &= \nabla \left(\sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes Sx^m \right) = \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} Sx^m. \end{aligned}$$

This is a linear system of equations that uniquely determines Sx^k by induction on k . Since

$$\sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q (-1)^m q^{m(m-1)/2} = \delta_{k,0} \quad (80)$$

Equation (75) follows. Finally (76) is clear since ϕ_t is an automorphism and $\phi_t x = tx$. \square

6.2. The left and right Yetter–Drinfel’d f-modules. In this section we compute the R -matrices of Theorem 3.6 explicitly. We first compute the doubly iterated coproduct (16), the coaction (17) and the R -matrix (19). The formulas involve the q -multinomial coefficients defined by

$$\begin{bmatrix} k \\ m, n \end{bmatrix}_q := \frac{(q; q)_k}{(q; q)_{k-m-n} (q; q)_m (q; q)_n}. \quad (81)$$

Lemma 6.2. The doubly iterated coproduct $\Delta^{(2)}$, the coaction δ and the R -matrix (19) ${}^L\rho$ are given by

$$\Delta^{(2)} x^k = \sum_{m=0}^k \sum_{n=0}^{k-m} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n \quad (82)$$

$$\delta x^k = \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m; q)_{k-m} x^{k-m} \otimes x^m \quad (83)$$

$${}^L\rho(x^k \otimes x^l) = \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^{k-m}; q)_m (tq^{k-m})^l x^{l+m} \otimes x^{k-m}. \quad (84)$$

Proof. We compute

$$\begin{aligned} \Delta^{(2)} x^k &= \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes \Delta x^m = \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes \sum_{n=0}^m \begin{bmatrix} m \\ n \end{bmatrix}_q x^{m-n} \otimes x^n \\ &= \sum_{0 \leq n \leq m \leq k} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} m \\ n \end{bmatrix}_q x^{k-m} \otimes x^{m-n} \otimes x^n \\ &= \sum_{n=0}^k \sum_{m=0}^{k-n} \begin{bmatrix} k \\ m+n \end{bmatrix}_q \begin{bmatrix} m+n \\ n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n \\ &= \sum_{n=0}^k \sum_{m=0}^{k-n} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n = \sum_{m=0}^k \sum_{n=0}^{k-m} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n \end{aligned} \quad (85)$$

and, using Equations (76) and (75),

$$\begin{aligned}
\delta x^k &= \sum_{m=0}^k \sum_{n=0}^{k-m} \begin{bmatrix} k \\ m, n \end{bmatrix}_q (\nabla \otimes \text{id}_H)(\text{id}_H \otimes \tau)(x^{k-m-n} \otimes x^m \otimes S\phi_t x^n) \\
&= \sum_{m=0}^k \sum_{n=0}^{k-m} q^{mn} \begin{bmatrix} k \\ m, n \end{bmatrix}_q (\nabla \otimes \text{id}_H)(x^{k-m-n} \otimes S\phi_t x^n \otimes x^m) \\
&= \sum_{m=0}^k \sum_{n=0}^{k-m} q^{mn} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m-n} S\phi_t x^n \otimes x^m \\
&= \sum_{m=0}^k \sum_{n=0}^{k-m} q^{mn} (-t)^n q^{n(n-1)/2} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m} \otimes x^m \\
&= \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q \sum_{n=0}^{k-m} \begin{bmatrix} k-m \\ n \end{bmatrix}_q (-tq^m)^n q^{n(n-1)/2} x^{k-m} \otimes x^m \\
&= \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m; q)_{k-m} x^{k-m} \otimes x^m
\end{aligned} \tag{86}$$

where the last equality follows from the q -binomial theorem. Thus, the R -matrix (19) for $Y = H$ is given by

$$\begin{aligned}
{}^L\rho(x^k \otimes x^l) &= (\nabla \otimes \text{id}_H)(\text{id}_H \otimes \tau_{H,H})(\delta \otimes \phi_H)(x^k \otimes x^l) \\
&= \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m; q)_{k-m} t^l (\nabla \otimes \text{id}_H)(\text{id}_H \otimes \tau_{H,H})(x^{k-m} \otimes x^m \otimes x^l) \\
&= \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m; q)_{k-m} (tq^m)^l x^{k+l-m} \otimes x^m \\
&= \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^{k-m}; q)_m (tq^{k-m})^l x^{l+m} \otimes x^{k-m}.
\end{aligned} \tag{87}$$

□

We next compute the doubly iterated product (16), the right action (18) and the R -matrix (20).

Lemma 6.3. The doubly iterated product $\nabla^{(2)}$, the action δ and the R -matrix (19) ρ are given by

$$\nabla^{(2)}(x^k \otimes x^l \otimes x^m) = x^{k+l+m} \tag{88}$$

$$\lambda(x^k \otimes x^l) = (tq^k; q)_l x^{k+l} \tag{89}$$

$$\rho^R(x^k \otimes x^l) = \sum_{m=0}^l \begin{bmatrix} l \\ m \end{bmatrix}_q (tq^k)^{l-m} (tq^k; q)_m x^{l-m} \otimes x^{k+m}. \tag{90}$$

Proof. Equation (88) is clear. To calculate the right action $\lambda(x^k \otimes x^l)$, we start with the case $l = 1$:

$$\begin{aligned} \lambda(x^k \otimes x) &= \nabla^{(2)}(S\phi_t \otimes \text{id}_{H \otimes H})(\tau \otimes \text{id}_G)(x^k \otimes x \otimes 1 + x^k \otimes 1 \otimes x) \\ &= \nabla^{(2)}(S\phi_t \otimes \text{id}_{H \otimes H})(q^k x \otimes x^k \otimes 1 + 1 \otimes x^k \otimes x) \\ &= \nabla^{(2)}(-tq^k x \otimes x^k \otimes 1 + 1 \otimes x^k \otimes x) = -tq^k x^{k+1} + x^{k+1} \\ &= (1 - tq^k)x^{k+1}. \end{aligned} \tag{91}$$

Now, we have

$$\begin{aligned} \lambda(x^k \otimes x^l) &= \lambda(\lambda(x^k \otimes x) \otimes x^{l-1}) = (1 - tq^k)\lambda(x^{k+1} \otimes x^{l-1}) \\ &= (1 - tq^k)(1 - tq^{k+1})\lambda(x^{k+2} \otimes x^{l-2}) = \dots = (tq^k; q)_k \lambda(x^{k+l} \otimes x^{l-l}) \\ &= (tq^k; q)_l x^{k+l}. \end{aligned} \tag{92}$$

Thus, the R -matrix (20) for $Y = H$ is given by

$$\begin{aligned} \rho^R(x^k \otimes x^l) &= \sum_{m=0}^l \begin{bmatrix} l \\ m \end{bmatrix}_q (\phi_H \otimes \lambda)(\tau_{H,H} \otimes \text{id}_H)(x^k \otimes x^m \otimes x^{l-m}) \\ &= \sum_{m=0}^l \begin{bmatrix} l \\ m \end{bmatrix}_q q^{km} t^m (tq^k; q)_{l-m} x^m \otimes x^{k+l-m} \\ &= \sum_{m=0}^l \begin{bmatrix} l \\ m \end{bmatrix}_q (tq^k)^{l-m} (tq^k; q)_m x^{l-m} \otimes x^{k+m}. \end{aligned} \tag{93}$$

□

The R -matrices (84) and (90) depend on two variables t and q , and using the basis $B = \{x^k \mid k \in \mathbb{Z}_{\geq 0}\}$, their entries are in $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ and satisfy the Yang–Baxter equation on an infinite dimensional space $T(\mathbb{F})$.

However, to define knot invariants as state-sums, we need to have rigid R -matrices over finite dimensional vector spaces. In the remaining subsections we give several solutions to this problem and identify the corresponding knot invariants.

6.3. Finite dimensional Nichols f-algebra: the ADO polynomials. The Nichols f-algebra $\mathfrak{B}(\mathbb{F})$ is finite-dimensional if and only if q is a root of unity of order $N > 1$. Indeed, when q is a root of unity of order $N \in \mathbb{Z}_{>1}$, it follows that $\begin{bmatrix} N \\ k \end{bmatrix}_q = 0$ for $0 < k < N$ and Equation (74) implies that x^N is primitive and thus generates a Hopf ideal of $\mathbb{F}[x]$ with finite-dimensional Nichols algebra $\mathbb{F}[x]/(x^N)$. (The converse is also true).

In this case, the R -matrix (84) coincides with the R -matrix of Akutsu–Deguchi–Ohtsuki [ADO92] and the knot invariant of Theorem 3.7 is the ADO polynomial times the identity matrix.

Remark 6.4. We remark that when $q = 1$, the Nichols algebra $\mathfrak{B}(\mathbb{F}) = \mathbb{F}[x]$ is not finite dimensional, thus we will exclude this classical value from our consideration of polynomial invariants. In fact, in this case one can still define invariants of knots using unipotent endomorphisms $\phi_t x = tx$ (i.e., assuming that $t - 1$ is nilpotent), and the corresponding knot

invariant is the inverse Alexander polynomial $1/\Delta(t)$, or else one can use a ground field \mathbb{F} of finite characteristic.

6.4. Finite dimensional Yetter–Drinfel’d f-modules: colored Jones polynomials.

When q is not a root of unity, the Nichols f-algebra $\mathfrak{B}(\mathbb{F}) = \mathbb{F}[x]$ is infinite-dimensional. However, it turns out that one can extract finite dimensional left or right Yetter–Drinfel’d f-modules if

$$t = q^{1-N}, \quad N \in \mathbb{Z}_{>0}. \quad (94)$$

Indeed, under this assumption for the scaling automorphism, Equation (83), implies that x^N is a coinvariant element, $\delta x^N = 1 \otimes x^N$. This gives a quotient left Yetter–Drinfel’d f-module $\mathbb{F}[x]/(x^N)$ of dimension N . The corresponding R -matrix is the one of the N -colored Jones polynomial.

Moreover, under the assumption (94), Equation (91) implies that x^{N-1} is an invariant element, whose coproduct generates the N -dimensional space with basis x^k , $0 \leq k \leq N-1$, and this gives an N -dimensional right Yetter–Drinfel’d f-submodule of $\mathbb{F}[x]$. The corresponding knot invariant is the identity matrix times the N -th colored Jones polynomial.

Summarising, in the rank 1 case, the corresponding matrix-valued knot invariants are the identity times the ADO and the colored Jones polynomials.

7. A RANK 2 TENSOR ALGEBRA

In this section we discuss the case of Nichols f-algebras of rank 2 of diagonal type. In order to keep the construction as simple as possible, we consider the Nichols f-algebra $\mathfrak{B}(V)$ associated with two-dimensional vector space V with basis $B = \{x_1, x_2\}$ and diagonal braiding

$$\tau(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad (q_{ij}) = \begin{pmatrix} -1 & q_{12} \\ q_{21} & -1 \end{pmatrix}. \quad (95)$$

In Heckenberger’s list [Hec08, Table 1] the isomorphism type of this Nichols algebra is determined by the parameter $q := q_{12}q_{21}$ of the generalized Dynkin diagram (see also [Hec07, Defn.3.1]).

For generic values of q , this Nichols algebra, which we denote \tilde{H}_q , is infinite-dimensional, and it is presented by the relations $x_1^2 = x_2^2 = 0$, so that a basis of it is given by alternating words in letters x_1 and x_2 .

7.1. q a root of unity: two-variable knot polynomials. When $q = \omega$ is a root of unity of order $N > 1$, the element $c_N := (x_2x_1)^N + (-q_{21}x_1x_2)^N$ is primitive and thus generates a Hopf ideal. The corresponding quotient Hopf algebra is a $4N$ -dimensional Nichols algebra which we denote $H_\omega = \tilde{H}_\omega/I$, where $I := \tilde{H}_\omega c_N \tilde{H}_\omega$. By taking the scaling automorphism ϕ_t of H_ω defined by $\phi_t x_i = t_i x_i$ for $i = 1, 2$ where t_1 and t_2 are two indeterminates, we obtain two-parameter R -matrices (not counting q_{12} which is irrelevant). The corresponding Nichols f-algebra denoted as $H_{\omega,t}$, considered as a left Yetter–Drinfel’d f-module over itself gives rise to an associated invariant $W_K^{H_{\omega,t}} \in \text{End}(H_{\omega,t})$ of a knot K illustrating Theorem 3.7. Let $Q_K(\omega, t_1, t_2) \in \mathbb{Z}[\omega, t_1^{\pm 1}, t_2^{\pm 1}]$ denote the $(1, 1)$ -entry of $W_K^{H_{\omega,t}}$.

Conjecture 7.1. For every knot K , we have

$$W_K^{H_{\omega,t}}(t_1, t_2) = Q_K(\omega, t_1, t_2) \text{id}_{H_{\omega,t}}. \quad (96)$$

7.2. $\omega = -1$: the Harper polynomial? In this section we discuss in detail the 2-parameter family of knot invariants defined when $N = 2$ (thus $\omega = -1$). We give examples of computations, compare our invariant with a known one, and list some conjectures regarding its structure.

When $\omega = -1$, the Nichols f-algebra $H_{-1,t}$ is 8-dimensional and isomorphic to the nilpotent Borel subalgebra of the small quantum group $u_q(\mathfrak{sl}_3)$ with $q = \sqrt{-1}$. A basis for $H_{-1,t}$ is

$$\{1, x_1, x_2, x_1x_2, x_2x_1, x_1x_2x_1, x_2x_1x_2, x_1x_2x_1x_2\}$$

and the corresponding 64×64 R -matrix has entries in $\mathbb{Z}[t_1, t_2, q_{12}^{\pm 1}]$ and has been computed explicitly. It is a sparse matrix with 3939 zero entries, and the remaining 157 entries nonzero. Due to its size, we do not present these entries here, but give a sample value

$$\begin{aligned} R(x_1x_2x_1x_2 \otimes x_2x_1) &= s^2t^2 x_2x_1 \otimes x_1x_2x_1x_2 - q_{21}^{-1}s^2t(1+t) x_1x_2x_1 \otimes x_2x_1x_2 \\ &\quad + q_{21}^2(-1+s)st^2 x_2x_1x_2 \otimes x_1x_2x_1 + q_{21}^{-1}(-1+s)st x_1x_2x_1x_2 \otimes x_2x_1, \end{aligned} \quad (97)$$

where for simplicity we abbreviate t_1 and t_2 by s and t .

For all knots for which we computed the invariant, we confirmed that Conjecture 7.1 holds, and what is more, the Laurent polynomial $Q_K(-1, s^2, t^2)$ coincides with Harper's polynomial $\Delta_{\mathfrak{sl}_3}(s, t)$ [Har].

Experimentally, it appears that the polynomial $Q_K(-1, t, s)$ is invariant under the involution maps

$$(s, t) \mapsto (t, s), \quad (s, t) \mapsto (s, -1/(st)), \quad (s, t) \mapsto (1/s, 1/t) \quad (98)$$

which generate a group G of order 12. The invariant polynomial ring can be identified with

$$\mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}]^G = \mathbb{Q}[u, v], \quad (99)$$

where

$$u = \langle s \rangle + \langle t \rangle - \langle st \rangle - 2, \quad v = \langle s^2t \rangle + \langle st^2 \rangle - \langle s/t \rangle - 2, \quad \langle x \rangle = x + x^{-1}. \quad (100)$$

In fact, we found that the polynomial Q_K is of the form

$$Q_K(-1, s, t) = P_K(u, v), \quad P_K(u, v) \in \mathbb{Z}[u, v]. \quad (101)$$

We further found that the polynomial Q_K does not distinguish a knot from its mirror image, and the special case $u = 0$ reproduces the Alexander–Conway polynomial $\nabla_K(z)$

$$P_K(0, z^2) = \nabla_K(z). \quad (102)$$

Table 1 gives the result of computer calculation for all knots of up to 6 crossings, and few higher crossing knots.

SnapPy confirms that the knots 7_4 and $\overline{9_2}$ (where \overline{K} is the mirror image of K) have equal Knot Floer Homology (a well-known fact; see Manolescu [Man16]), thus have Seifert genus 1 and none is fibered [CDGW]. On the other hand, the two knots have different Q -polynomial; see Table 1.

The colored Jones and the ADO polynomials do not distinguish mutant pairs of knots, since the corresponding tensor product of representations is multiplicity-free. This fact was pointed out to us by J. Murakami and T. Ohtsuki. On the other hand, the Q -polynomial

detects mutation, and sometimes not. In particular, it distinguishes the mutant Kinoshita–Terasaka ($11n_{42}$) and Conway ($11n_{34}$) pair of knots—a fact that was pointed out by Harper in [Har] for his polynomial $\Delta_{\mathfrak{sl}_3}(s, t)$.

On the other hand, unlike Knot Floer Homology, it does not distinguish the mutant pairs $11n_{74}$ (a fibered, Seifert genus 2 knot) and $11n_{73}$ (a non-fibered, Seifert genus 3 knot).

Finally, the knots 8_8 and 10_{129} have isomorphic Khovanov homology [BN02], yet different Q -polynomial.

Knot K	The polynomial $P_K(u, v)$
3_1	$1 + 4u + u^2 + v$
4_1	$1 - 6u + u^2 - v$
5_1	$1 + 12u + 19u^2 + 8u^3 + u^4 + (3 + 7u + 3u^2)v + v^2$
5_2	$1 + 10u + 6u^2 + 2v$
6_1	$1 - 10u + 6u^2 - 2v$
6_2	$1 - 8u - 15u^2 + 2u^3 + u^4 + (-1 - 9u + u^2)v - v^2$
6_3	$1 + 2u + 15u^2 + 6u^3 + u^4 + (1 + 9u + u^2)v + v^2$
7_1	$1 + 24u + 86u^2 + 104u^3 + 53u^4 + 12u^5 + u^6$ $+ (6 + 35u + 60u^2 + 33u^3 + 5u^4)v + (5 + 10u + 6u^2)v^2 + v^3$
7_4	$(1 + 2u)(1 + 18u) + 4v$
8_8	$1 + 10u + 36u^2 + 28u^3 + 6u^4 + 2(1 + 9u + 3u^2)v + 2v^2$
8_{17}	$1 - 14u - 23u^2 - 38u^3 + 10u^4 + 8u^5 + u^6$ $+ (-1 - 17u - 44u^2 + 5u^3 + 2u^4)v + (-2 - 13u + u^2)v^2 - v^3$
9_2	$1 + 20u + 24u^2 + 4v$
10_2	$1 - 150u^2 - 380u^3 - 279u^4 - 44u^5 + 25u^6 + 10u^7 + u^8$ $+ (2 - 55u - 260u^2 - 274u^3 - 58u^4 + 19u^5 + 5u^6)v$ $+ (-5 - 62u - 91u^2 - 24u^3 + 5u^4)v^2 + (-5 - 15u - 2u^2)v^3 - v^4$
10_{129}	$1 + 10u + 32u^2 + 36u^3 + 6u^4 + 2(1 + 8u + 2u^2)v + 2v^2$
$11n_{34}$	$1 + 12u + 8u^2 + 60u^3 + 48u^4 + 8u^5 + 2u(1 + 2u)(-1 + 6u)v + 2u^2v^2$
$11n_{42}$	$1 + 12u + 8u^2 - 12u^3 - 2uv$
$11n_{73}$ $11n_{74}$	$1 + 20u + 10u^2 + 4u^3 + u^4 + 2(1 + 4u + u^2)v + v^2$

TABLE 1. The polynomial $P_K(u, v)$ for some knots

7.3. q generic: two-variable polynomials. In this section we classify all finite-dimensional right Yetter–Drinfel’d \mathfrak{f} -modules by classifying all maximal vectors. Denote by $\tilde{H}_{q,t}$ the Nichols \mathfrak{f} -algebra \tilde{H}_q with a chosen scaling automorphism ϕ_t . Recall that it has a basis that consists of all alternating words in the letters x_1 and x_2 , where $x_1^2 = x_2^2 = 0$. Thus, every basis element is of the form

$$(x_1x_2)^a, \quad \text{or} \quad (x_2x_1)^b, \quad \text{or} \quad x_2(x_1x_2)^c, \quad \text{or} \quad x_1(x_2x_1)^d \quad (103)$$

for a unique $a, b, c, d \in \mathbb{Z}_{\geq 0}$. Moreover, $\tilde{H}_{q,t}$ is $\mathbb{Z}_{\geq 0}^2$ -graded, thus also $\mathbb{Z}_{\geq 0}$ -graded where the $\mathbb{Z}_{\geq 0}$ -degree is the sum of the components of the $\mathbb{Z}_{\geq 0}^2$ -bi-degree.

It follows from (103) that the degree $2n - 1$ part of $\tilde{H}_{q,t}$ is the direct sum of two bi-degrees $(n, n - 1)$ and $(n - 1, n)$, each of them being one-dimensional. This implies that there are no invariant vectors of degree $2n - 1$. Indeed, the only vector of bi-degree $(n - 1, n)$ is

$$x := (x_2 x_1)^{n-1} x_2 = x_2 (x_1 x_2)^{n-1}$$

which is λ -annihilated by x_2 but not by x_1 :

$$\lambda(x \otimes x_1) = (x_2 x_1)^n + (-q_{21})^n t_1 (x_1 x_2)^n \quad (104)$$

which never vanishes since the vectors $(x_2 x_1)^n$ and $(x_1 x_2)^n$ are linearly independent.

Thus, invariant vectors can only be of even degree $2n$ which corresponds to bi-degree (n, n) . The corresponding subspace is two-dimensional with the basis vectors $(x_1 x_2)^n$ and $(x_2 x_1)^n$. Taking a vector of the form

$$v_{n,\alpha} = (x_1 x_2)^n + \alpha (x_2 x_1)^n, \quad \alpha \in \mathbb{F}, \quad (105)$$

we calculate the λ -action on it of the generating elements:

$$\lambda(v_{n,\alpha} \otimes x_1) = (1 - \alpha t_1 (-q_{21})^n) (x_1 x_2)^n x_1 \quad \lambda(v_{n,\alpha} \otimes x_2) = (\alpha - t_2 (-q_{12})^n) (x_2 x_1)^n x_2. \quad (106)$$

Thus, $v_{n,\alpha}$ is an invariant vector if and only if

$$\alpha = t_2 (-q_{12})^n, \quad t_1 t_2 q_{12}^n = 1. \quad (107)$$

Proposition 7.2. Assume that q is not a root of unity, and let the parameters α, t_1, t_2 satisfy relations (107) and the inequality $(1 - t_1)(1 - t_2) \neq 0$. Then, the right Yetter–Drinfel’d f-module Y_n generated by the element $v_{n,\alpha}$ defined in (105) is $4n$ -dimensional and it is the linear span of the vector $v_{n,\alpha}$ and all vectors of degree less or equal to $2n - 1$.

Proof. The coproduct of $v_{n,\alpha}$ always contains the term $1 \otimes v_{n,\alpha}$ so that $1 \in Y_n$. The λ -action on 1 gives

$$\lambda(1 \otimes x_i) = (1 - t_i) x_i, \quad i = 1, 2. \quad (108)$$

By the assumption on t_1 and t_2 , we conclude that vectors x_1 and x_2 are both in Y_n . Assume by induction that both vectors in odd degree $2k - 1$ are contained in Y_n where $1 \leq k < n$. Then, the λ -action on them of the generating elements produces two vectors in degree $2k$

$$\lambda((x_1 x_2)^{k-1} x_1 \otimes x_2) = (x_1 x_2)^k + t_2 (-q_{12})^k (x_2 x_1)^k \quad (109)$$

and

$$\lambda((x_2 x_1)^{k-1} x_2 \otimes x_1) = (x_2 x_1)^k + t_1 (-q_{21})^k (x_1 x_2)^k \quad (110)$$

which are linearly independent provided $t_1 t_2 q^k \neq 1$. Thus, the λ -action of the generating elements on all vectors in degree $2k$ produces all vectors in degree $2k + 1$. We conclude that all vectors in degree $\leq 2n - 1$ are in Y_n . Now, equations (109) and (110) at $k = n$ imply that both vectors are proportional to $v_{n,\alpha}$. \square

Thus, for any $n \in \mathbb{Z}_{\geq 1}$ we obtain an R -matrix over the $4n$ -dimensional vector space Y_n which according to Theorem 3.7 produces a knot invariant $W_K^{Y_n}(q, t_1) \in \text{End}(Y_n)$.

Conjecture 7.3. For every knot K , we have

$$W_K^{Y_n}(q, t_1) = Q_{n,K}(q, t_1) \text{id}_{Y_n}, \quad Q_{n,K}(q, t_1) \in \mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}]. \quad (111)$$

Calculations for $n = 1$ indicate that Conjecture 7.3 holds true and $Q_{1,K}(q, t_1)$ coincides with the Links–Gould two-variable knot polynomial coming from the quantum superalgebra $U_q(\mathfrak{gl}(2|1))$ [LG92].

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