q-TERMS, SINGULARITIES AND THE EXTENDED BLOCH GROUP

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Dedicated to S. Bloch on the occasion of his sixtieth birthday.

Abstract. Our paper originated from a generalization of the Volume Conjecture to multisums of q-hypergeometric terms. This generalization was sketched by Kontsevich in a problem list in Aarhus University in 2006; [Ko]. We introduce the notion of a special and general q-hypergeometric term (in short, special q-termand general q-term). The latter is a product of q-binomials and q-factorials in linear forms in several variables.

In the first part of the paper, using elementary manipulations of symbols, we show how to assign elements of the Bloch group to a general q-term. The image of these elements under the Bloch-Wigner regulator map is always a finite subset of the set of purely imaginary periods, in the sense of Kontsevich-Zagier.

In the second part of the paper, we extend our results to the extended Bloch group. The latter captures exactly the torsion information of $K_3^{\mathrm{ind}}(\mathbb{C})$, and its regulator is given by the Rogers dilogarithm function. Our outcome is again a finite subset of the set of periods.

In the third part of the paper, given a special q-term, we can associate two power series, convergent in a neighborhood of zero. We conjecture that the series have analytic continuation as a multivalued function in the complex numbers minus the above described finite set of points. Our conjecture implies a strong form of the Volume Conjecture and is known to be true in case of 1-dimensional special q-terms, as follows from joint work with O. Costin.

In the final part of the paper, we describe a rich source of special terms that come from real 3-dimensional knotted objects. Finally we compare our combinatorial encodings of general q-terms with that of Neumann-Zagier and Kontsevich.

Contents

Date: March 2, 2007.
The author was supported in part by NSF.

1991 *Mathematics Classification.* Primary 57N10. Secondary 57M25.
Key words and phrases: Bloch group, extended Bloch group, algebraic K-theory, regulators, dilogarithm, Rogers dilogarithm, Bloch-Wigner dilogarithm, potential, Bethe ansatz, special q-hypergeometric terms, asymptotic expansions, knots, homology spheres, Habiro ring, Volume Conjecture, periods, Quantum Topology, resurgence, Gevrey series.

1. Introduction

1.1. **A brief summary of our results.** Our paper originated from a generalization of the Volume Conjecture to multisums of q-hypergeometric terms. This generalization was sketched by Kontsevich in a problem list in Aarhus University in 2006; [Ko]. Our paper expands Kontsevich's problem, and reveals a close and precise relation between special q-hypergeometric terms (defined below), elements of the extended Bloch group and its conjectural relation to singularities of generating series. Oddly enough, setting $q=1$ also provides a relation between special hypergeometric terms and elements of the *extended additive* Bloch group. This is explained in a separate publication; see [Ga1] and [Ga2].

As a guide to our results, we introduce the notion of a q-term below. Given a general q-term $\mathfrak{t}$, in the first part of the paper we will construct:

(a) A set of Variational Equations (see Definition 1.4) with solution set $X_{\mathfrak{t}}$ which is typically a finite set of solutions, given by algebraic numbers.
(b) A map:

$$\beta_{\mathfrak{t}} : X_{\mathfrak{t}} \longrightarrow \mathcal{B}(\mathbb{C})$$

where $\mathcal{B}(\mathbb{C})$ is the Bloch group; see Definition 1.3 and Theorem 1.
(c) The image of $\beta_{\mathfrak{t}}$ under the Bloch-Wigner regulator map is a finite subset of $i\mathbb{R}\cap\mathcal{P}$, where $\mathcal{P}$ is the set of periods in the sense of Kontsevich-Zagier; see [KZ].

Among other things, the map $\beta_{\mathfrak{t}}$ gives an explicit method for constructing elements of the Bloch group $\mathcal{B}(\mathbb{C})$. Assuming some standard conjectures, the map $\beta_{\mathfrak{t}}$ may construct all elements in the Bloch group $\mathcal{B}(\mathbb{C})$; see Section 6.3.

1.2. **What is a q-term?** The next definition plays a key role in our paper.

Definition 1.1. (a) A *special q-hypergeometric term* $\mathfrak{t}_{n,k}$ (in short, *special q-term* $\mathfrak{t}$) in variables n and $k=(k_1,\dots,k_r)$ is an expression of the form

$$\mathfrak{t}_{n,k}(q) = q^{Q(n,k)}\epsilon^{L(n,k)}\prod_{j=1}^{J}\binom{B_j(n,k)}{C_j(n,k)}_q (q)_{D_j(n,k)}, \tag{1}$$

where B_j, C_j, D_j and L are integral linear forms in the variables n, k, where $k = (k_1, \dots, k_r)$ is a multi-index, Q is an integral quadratic form Q and $\epsilon = \pm 1$.
(b) A *general q-term* $\mathfrak{t}_{n,k}$ in variables n and $k = (k_1, \dots, k_r)$ is an expression of the form

$$\mathfrak{t}_{n,k}(q) = q^{Q(n,k)} \epsilon^{L(n,k)} \prod_{j=1}^{J} (q)_{A_j(n,k)}^{\epsilon_j}. \tag{2}$$

where A_j and L are integral linear forms, Q is an integral quadratic form and $\epsilon = \pm 1$, $\epsilon_j = \pm 1$ for $j = 1, \dots, J$.

Here and throughout, the *q-factorial* and the *q-binomial* are defined by:

$$(q)_m = \prod_{j=1}^{m} (1 - q^j), \qquad \binom{m}{l}_q = \frac{(q)_m}{(q)_l (q)_{m-l}} \tag{3}$$

We will assume that for every $n \in \mathbb{N}$, we have $\mathfrak{t}_{n,k}(q) = 0$ for large enough k.

Remark 1.2. It is easy to see that every special q-term is a general q-term; see Lemma 7.1 from Section 7.1.
An alternative way of encoding a general q-term is to record ϵ, the $(r+1) \times (r+1)$ matrix of coefficients of Q, and list the coefficients of the linear forms $A_j(n, k)$ and $L(n, k)$ for $j = 1, \dots, J$ as a $(J + 1) \times (r + 1)$ matrix, where $k = (k_1, \dots, k_r)$. When $\epsilon = 1$, we can ignore $L(n, k)$. This encoding is very much in the spirit of Neumann-Zagier and Kontsevich; see [NZ, Ko] and also Section 7.

1.3. The Bloch group.

Let us recall the symbolic-definition of the Bloch group from [B1]; see also [DS, Su1, Su2].

Definition 1.3. (a) Let $\mathcal{P}(F)$ denote the abelian group generated by symbols $[z]$, $z \in F \setminus \{0, 1\}$, subject to the *5-term relation*:

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] + \left[\frac{1 - x}{1 - y}\right] = 0. \tag{4}$$

(b) The *Bloch group* $\mathcal{B}(F)$ is the kernel of the homomorphism

$$\nu : \mathcal{P}(F) \longrightarrow F^* \wedge F^*, \qquad [z] \mapsto z \wedge (1 - z) \tag{5}$$

to the second exterior power of the abelian group F^* defined by mapping a generator $[z]$ to $z \wedge (1 - z)$. The second exterior power $G \wedge G$ of an abelian group G is defined by:

$$G \wedge G = G \otimes_{\mathbb{Z}} G / (a \otimes b + b \otimes a) \tag{6}$$

(c) The 2-term complex (5) is the famous *Bloch-Suslin* complex that has motivated a lot of research.

Recall the *Bloch-Wigner function*

$$\mathcal{D}_2 : \mathbb{C} \longrightarrow i\mathbb{R}, \qquad z \mapsto \mathcal{D}_2(z) := i\Im(\mathrm{Li}_2(z) + \log(1 - z) \log |z|), \tag{7}$$

which is continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus \{0, 1\}$. Since $\mathcal{D}_2$ satisfies the 5-term relation, we can define a *regulator map* on the pre-Bloch group of the complex numbers:

$$R : \mathcal{P}(\mathbb{C}) \longrightarrow i\mathbb{R}, \qquad z \mapsto \mathcal{D}_2([z]) \tag{8}$$

and its restriction to the Bloch group (denoted by the same notation):

$$R : \mathcal{B}(\mathbb{C}) \longrightarrow i\mathbb{R}, \qquad z \mapsto \mathcal{D}_2([z]) \tag{9}$$

1.4. **From q-terms to the Bloch group.** In order to state our results, let us introduce some useful notation. Given a linear form A and $i = 1, \dots, r$, in variables (n, k) where $k = (k_1, \dots, k_r)$ let us define

$$v_i(A) = a_i, \qquad v_0(A) = a_0, \qquad \text{where} \qquad A(n,k) = a_0 n + \sum_{i=1}^r a_i k_i. \tag{10}$$

For an integral linear form A on n, k let us abbreviate

$$z^A = \prod_{i=0}^r z_i^{v_i(A)}. \tag{11}$$

The next definition associates an affine scheme over $\mathbb{Q}$ to a general q-term $\mathfrak{t}$.

Definition 1.4. Given a general q-term $\mathfrak{t}$ as in (2), consider the set $X_{\mathfrak{t}}$ of points $z = (z_0, \dots, z_r)$ that satisfy the following system of *Variational Equations*:

$$z^{\frac{\partial Q}{\partial z_i}} \epsilon^{v_i(L)} \prod_{j=1}^J (1 - z^{A_j})^{v_i(\epsilon_j A_j)} = 1 \tag{12}$$

for $i = 0, \dots, r$. In particular, z lies in the set of complex points of an affine scheme defined over $\mathbb{Q}$.

Typically, one expects that the Variational Equations (12) have a finite set of complex solutions. In that case, the solutions lie in $\overline{\mathbb{Q}}$. The Variational Equations (12) are reminiscent of the *Bethe ansatz* in quantum field theory. The next definition assigns elements of the pre-Bloch group to a general q-term $\mathfrak{t}$.

Definition 1.5. With the above conventions, consider the map:

$$\beta_{\mathfrak{t}} : X_{\mathfrak{t}} \longrightarrow \mathcal{P}(\mathbb{C}) \tag{13}$$

given by:

$$z \mapsto \beta_{\mathfrak{t}}(z) := \sum_{j=1}^{4J} \epsilon_j [z^{A_j}]. \tag{14}$$

The next theorem, communicated to us by Kontsevich, assigns elements of the Bloch group to a general q-term $\mathfrak{t}$.

Theorem 1. (a) The map $\beta_{\mathfrak{t}}$ descends to a map

$$\beta_{\mathfrak{t}} : X_{\mathfrak{t}} \longrightarrow \mathcal{B}(\mathbb{C}) \tag{15}$$

which we denote by the same name.
(b) The image of

$$R \circ \beta_{\mathfrak{t}} : X_{\mathfrak{t}} \longrightarrow i\mathbb{R}$$

is a finite subset of $i\mathbb{R} \cap \mathcal{P}$, where $\mathcal{P}$ is the set of periods in the sense of Kontsevich-Zagier; [KZ].

1.5. **An example.** To better understand the content of Theorem 1, let us give a simple example, which is already nontrivial and of interest to Quantum Topology. Consider the general q-term

$$\mathfrak{t}_n(q) = q^{a\frac{n(n+1)}{2}} \epsilon^n (q)_n^b \tag{16}$$

where $a \in \mathbb{Z}$, $\epsilon = \pm 1$ and $b \in \mathbb{N}$. In other words, $k = \emptyset$. The Variational Equations (12) become:

$$z^a (1-z)^b \epsilon = 1 \tag{17}$$

$X_{\mathfrak{t}}$ is the finite set of complex solutions of (17). The map (13) is given by:

$$X_{\mathfrak{t}} \longrightarrow \mathcal{B}(\mathbb{C}), \qquad z \mapsto \beta_{\mathfrak{t}}(z) := a[z]. \tag{18}$$

Curiously enough, Equation (17) and the corresponding element of the Bloch group was studied by Lewin, using his method of *ladders*. In that sense, the Variational Equations (12) is a generalization of the method of ladders.

Let us end this section with a remark.

Remark 1.6. Among other things, Theorem 2 is a practical way of constructing elements of the Bloch group $\mathcal{B}(\mathbb{C})$ that are typically defined over number fields. For examples, see Section 7. In general, $X_{\mathfrak{t}}$ is not 0-dimensional, although its image under $R \circ \beta_{\mathfrak{t}}$ always is finite. This finiteness is positive evidence for Bloch's *Rigidity Conjecture*, which states that

$$\mathcal{B}(\overline{\mathbb{Q}}) \otimes \mathbb{Q} \cong \mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}.$$

1.6. **Acknowledgement.** An early version of this paper was discussed with M. Kontsevich in the IHES and University of Miami and with S. Bloch in Chicago. The author wishes to thank S. Bloch and M. Kontsevich for generously sharing their ideas with the author and for their hospitality in the IHES in the fall of 2006 and in Chicago in the spring of 2007. The paper was finished in the spring of 2007, and updated further in the summer of 2007 to incorporate the final extension of the Rogers dilogarithm due to Goette and Zickert. The author wishes to thank D. Thurston and C. Zickert for many stimulating conversations.

2. Proof of Theorem 1

This section is devoted to a proof of Theorem 1, using elementary symbol-like manipulations.

Proof. (of Theorem 1) Let us fix $z = (z_0, \dots, z_r)$ that satisfies the Variational Equations (12). We will show that $\nu(\beta_{\mathfrak{t}}(z)) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*$, where ν is given by (5). We compute as follows:

$$\beta_{\mathfrak{t}}(z) \quad = \quad \sum_{j=1}^{J} \epsilon_j (z^{A_j} \wedge (1 - z^{A_j})).$$

On the other hand, we have:

$$z^{A_j} = \prod_{i=0}^{r} z_i^{v_i(A_j)}.$$

Thus,

$$\begin{aligned} \epsilon_j (z^{A_j} \wedge (1 - z^{A_j})) \quad &= \quad \epsilon_j \sum_{i=0}^{r} z_i^{v_i(A_j)} \wedge (1 - z^{A_j}) \\ &= \quad \sum_{i=0}^{r} \epsilon_j v_i(A_j) (z_i \wedge (1 - z^{A_j})) \\ &= \quad \sum_{i=0}^{r} z_i \wedge (1 - z^{A_j})^{v_i(\epsilon_j A_j)}. \end{aligned}$$

Since z satisfies the Variational Equations (12), after we interchange the j and i summation, we obtain that:

$$\begin{aligned}
\beta_{\mathfrak{t}}(z) &= \sum_{i=0}^{r}\sum_{j=1}^{J} z_i \wedge (1-z^{A_j})^{v_i(\epsilon_j A_j)} \\
&= \sum_{i=0}^{r} z_i \wedge \prod_{j=1}^{J}(1-z^{A_j})^{v_i(\epsilon_j A_j)} \\
&= \sum_{i=0}^{r} z_i \wedge (z^{-\frac{\partial Q}{\partial z_i}}\epsilon^{-v_i(L)}) \\
&= -\sum_{i=0}^{r} z_i \wedge z^{\frac{\partial Q}{\partial z_i}} - \sum_{i=0}^{r} z_i^{v_i(L)} \wedge \epsilon \\
&= -\sum_{i=0}^{r} z_i \wedge z^{\frac{\partial Q}{\partial z_i}} - z^{L} \wedge \epsilon.
\end{aligned}$$

Since Q is an integral symmetric bilinear form and $\wedge$ is skew-symmetric, it follows that

$$\sum_{i=0}^{r} z_i \wedge z^{\frac{\partial Q}{\partial z_i}} = 0.$$

In addition, we claim that for $\epsilon = \pm 1$, we have:

$$z \wedge \epsilon = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*. \tag{19}$$

Indeed, if $\epsilon = 1$, then

$$z \wedge 1 = z \wedge (1.1) = z \wedge 1 + z \wedge 1.$$

If $\epsilon = -1$, then

$$z^2 \wedge (-1) = 2(z \wedge (-1)) = z \wedge ((-1)^2) = z \wedge 1 = 0.$$

Since $\mathbb{C}^* = (\mathbb{C}^*)^2$, the result follows. Thus,

$$\nu(\beta_{\mathfrak{t}}(z)) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*.$$

This proves that $\beta_{\mathfrak{t}}$ takes values in the Bloch group, and concludes part (a) of Theorem 2.

Part (b) follows from a stronger result; see part (c) of Theorem 2 below. The idea is that any analytic function is constant on a connected component of its critical set. In our case, there exists a potential function whose points are the complex points of an affine variety defined over $\mathbb{Q}$ by the Variational Equations. The complex points of an affine variety have finitely many connected components. Finiteness of the image of $R \circ \beta_{\mathfrak{t}}$ follows. □

3. From q-terms to the extended Bloch group

In the remainder of the paper, we will extend our results from the Bloch group to the extended Bloch group. As a guide, given a general q-term $\mathfrak{t}$, we will assign

(a) A potential function $V_{\mathfrak{t}}$; see Definition 3.10.

(b) A set of Logarithmic Variational Equations (see (12)), which typically have a finite set of solutions, given by algebraic numbers. The variational equations may be identified with the critical points $\widehat{X_{\mathfrak{t}}}$ of the potential $V_{\mathfrak{t}}$; see Proposition 3.12.

(c) A map

$$\hat{\beta}_{\mathfrak{t}} : \widehat{X_{\mathfrak{t}}} \longrightarrow \widehat{\mathcal{B}(\mathbb{C})}$$

where $\widehat{\mathcal{B}(\mathbb{C})}$ is the extended Bloch group; see Definition 3.1 and Theorem 2. When $\hat{\beta}_{\mathfrak{t}}$ is composed with a regulator map $\hat{R}$ (given in Equation (30)), it essentially coincides with the evaluation of the potential on its critical values; see Theorem 2.

(d) The image of the composition $e^{\hat{R}/(2\pi i)} \circ \hat{\beta}_{\mathrm{t}}$ is a finite set of complex numbers, given by exponentials of periods; see Theorem 2.

In the last part of the paper, which was a motivation all along, we mention that quantum invariants of 3-dimensional knotted objects (or general statistical-mechanical sums) give rise to special q-terms, a fact that was initially observed in [GL1]. We also make connection between our Variational Equations and the Gluing Equations of Neumann-Zagier; see Lemma 7.2.

3.1. The extended Bloch group.

Our aim is to associate elements of the extended Bloch group to every general q-term. The extended Bloch group was introduced by Neumann in his investigation of the Cheeger-Chern-Simons classes and hyperbolic 3-manifolds; see [Ne]. The definition below is called the very-extended Bloch group by Neumann, see also [DZ].

There is a close relation between the Bloch group $\mathcal{B}(F)$ and $K_3^{\mathrm{ind}}(F)$, (indecomposable K-theory of F); see [B1, B2] and [Su1, Su2], as well as Section 3.3 below. Unfortunately, the relation among $\mathcal{B}(F)$ and $K_3^{\mathrm{ind}}(F)$ is known modulo torsion, and the torsion does not match. For $F = \mathbb{C}$, this is exactly what the extended Bloch group captures. The idea is that:

$$\boxed{\text{torsion is encoded by the choice of the branch of the logarithms}}.$$

More precisely, consider the doubly punctured plane

$$\mathbb{C}^{**} := \mathbb{C} \setminus \{0,1\} \tag{20}$$

and let $\hat{\mathbb{C}}$ denote the universal abelian cover of $\mathbb{C}^{**}$. We can represent the Riemann surface of $\mathbb{C}^{**}$ as follows. Let $\mathbb{C}_{\mathrm{cut}}$ denote $\mathbb{C}^{**}$ cut open along each of the intervals $(-\infty,0)$ and $(1,\infty)$ so that each real number r outside $[0,1]$ occurs twice in $\mathbb{C}_{\mathrm{cut}}$. Let us denote the two occurrences of r by $r+0i$ and $r-0i$ respectively. It is now easy to see that $\hat{\mathbb{C}}$ is isomorphic to the surface obtained from $\mathbb{C}_{\mathrm{cut}} \times 2\mathbb{Z} \times 2\mathbb{Z}$ by the following identifications:

$$\begin{array}{rcll} (x+0i;2p,2q) & \sim & (x-0i;2p+2,2q) & \text{for} \quad x \in (-\infty,0) \\ (x+0i;2p,2q) & \sim & (x-0i;2p,2q+2) & \text{for} \quad x \in (1,\infty) \end{array}$$

This means that points in $\hat{\mathbb{C}}$ are of the form (z,p,q) with $z \in \mathbb{C}^{**}$ and p,q even integers. Moreover, $\hat{\mathbb{C}}$ is the Riemann surface of the function

$$\mathbb{C}^{**} \longrightarrow \mathbb{C}^2, \qquad z \mapsto (\mathrm{Log}(z), \mathrm{Log}(1-z)).$$

where Log denotes the *principal branch* of the logarithm. Consider the set

$$\mathrm{FT} := \{(x, y, \frac{y}{x}, \frac{1-x^{-1}}{1-y^{-1}}, \frac{1-x}{1-y})\} \subset (\mathbb{C}^{**})^5$$

of 5-tuples involved in the 5-term relation. Also, let

$$\mathrm{FT}_0 := \{(x_0,\dots,x_4) \in \mathrm{FT} | 0 < x_1 < x_0 < 1\}$$

and define $\widehat{\mathrm{FT}} \subset \hat{\mathbb{C}}^5$ to be the component of the preimage of FT that contains all points $((x_0;0,0),\dots,(x_4;0,0))$ with $(x_0,\dots,x_4) \in \mathrm{FT}_0$. See also [DZ, Rem.2.1].

We can now define the extended Bloch group, following [GZ, Def.1.5].

Definition 3.1. (a) The *extended pre-Bloch group* $\widehat{\mathcal{P}(\mathbb{C})}$ is the abelian group generated by the symbols $[z;p,q]$ with $(z;p,q) \in \hat{\mathbb{C}}$, subject to the relation:

$$\sum_{i=0}^{4} (-1)^i [x_i;p_i,q_i] = 0 \quad \text{for} \quad ((x_0;p_0,q_0),\dots,(x_4;p_4,q_4)) \in \widehat{\mathrm{FT}}. \tag{21}$$

(b) The *extended Bloch group* $\widehat{\mathcal{B}(\mathbb{C})}$ is the kernel of the homomorphism

$$\hat{\nu} : \widehat{\mathcal{P}(\mathbb{C})} \longrightarrow \mathbb{C} \wedge \mathbb{C}, \qquad [z; p, q] \mapsto (\mathrm{Log}(z) + p\pi i) \wedge (-\mathrm{Log}(1-z) + q\pi i) \tag{22}$$

(c) We will call the 2-term complex (22) the *extended Bloch-Suslin* complex.

For a comparison between the extended Bloch-Suslin complex and the Suslin complex, see Section 8.3.

Remark 3.2. There are four versions of the extended Bloch group, depending on whether $(p,q) \in \mathbb{Z}^2$ versus $(p,q) \in (2\mathbb{Z})^2$, and whether we include the transfer relation of [Ne, Eqn.3]. For a comparison of Neumann's version of the extended Bloch group with the above definition, see [GZ, Rem.4.3].

3.2. **The Rogers dilogarithm.** The extended Bloch-Suslin complex (22) is very closely related to the functional properties of a normalized form of the dilogarithm. In fact, it would be hard to motivate the extended Bloch-Suslin complex without knowing the Rogers dilogarithm and vice-versa.

Following [Ne, DZ], the *Rogers dilogarithm* is the following function defined on the open interval $(0,1)$:

$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\mathrm{Log}(z)\mathrm{Log}(1-z) - \frac{\pi^2}{6} \tag{23}$$

where

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{Log}(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is the classical *dilogarithm function.* In [GZ, Defn.1.5], Goette and Zickert extended $L(z)$ to a multivalued analytic function on $\hat{\mathbb{C}}$ as follows (see [Ne, Prop.2.5]):

$$\hat{R} : \hat{\mathbb{C}} \longrightarrow \mathbb{C}/\mathbb{Z}(2) \tag{24}$$

$$\hat{R}(z;p,q) \quad := \quad \mathrm{Li}_2(z) + \frac{1}{2}(\mathrm{Log}(z) + \pi i p)(\mathrm{Log}(1-z) + \pi i q) - \frac{\pi^2}{6} \tag{25}$$

$$= \quad L(z) + \frac{\pi i}{2}(q\mathrm{Log}(z) + p\mathrm{Log}(1-z)) - \frac{\pi^2 pq}{2}. \tag{26}$$

where as usual, for a subgroup K of $(\mathbb{C}, +)$ and an integer $n \in \mathbb{Z}$, we denote $K(n) = (2\pi i)^n K \subset \mathbb{C}$.

Let us comment a bit on the properties of the Rogers dilogarithm.

Remark 3.3. A computation involving the monodromy of the Li_2 function shows that the dilogarithm Li_2 has an analytic extension:

$$\mathrm{Li}_2 : \hat{\mathbb{C}} \longrightarrow \mathbb{C}/\mathbb{Z}(2) \tag{27}$$

See [Oe, Prop.1]. Compare also with [Ne, Prop.2.5]. In [GZ, Lem.2.2] it is shown that $\hat{R}$ takes values in $\mathbb{C}/\mathbb{Z}(2)$ and that the Rogers dilogarithm is defined on the extended pre-Bloch group; we will denote the extension by $\hat{R}$. In other words, we have:

$$\hat{R} : \widehat{\mathcal{P}(\mathbb{C})} \longrightarrow \mathbb{C}/\mathbb{Z}(2). \tag{28}$$

In [DZ] and [GZ] it was shown that the Rogers dilogarithm coincides with the $\mathbb{C}/\mathbb{Z}(2)$-valued *Cheeger-Chern-Simons class* (denoted $\hat{C}_2$ in [DZ]). For a discussion on this matter, see [DZ, Thm.4.1].

Remark 3.4. It is natural to ask why to modify the dilogarithm as in Rogers extension. The motivation is that $\hat{L}$ (but *not* the dilogarithm function $\mathrm{Li}_2(z)$ nor the Bloch-Wigner dilogarithm) satisfies the extended 5-term relation $\widehat{\mathrm{FT}}$; see [DZ, Sec.2] and [GZ, Lem.2.2].

Remark 3.5. For the purposes of comparing the Rogers dilogarithm with other special functions (such as the entropy function discussed in [Ga1]) It is easy to see that the derivative of the Rogers dilogarithm is given by:

$$L'(z) = -\frac{1}{2}\left(\frac{\log(1-z)}{z} + \frac{\log(z)}{1-z}\right). \tag{29}$$

Thus, $\hat{R}$ descends to a *regulator map*:

$$\hat{R} : \widehat{\mathcal{B}(\mathbb{C})} \longrightarrow \mathbb{C}/\mathbb{Z}(2) \tag{30}$$

which can be exponentiated to a map:

$$e^{\frac{1}{2\pi i}\hat{R}} : \widehat{\mathcal{B}(\mathbb{C})} \longrightarrow \mathbb{C}^*. \tag{31}$$

The regulators R and $\hat{R}$ of the Bloch group and its extended version are part of the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathcal{B}(\mathbb{C})} \arrow[r, "\hat{R}"] \arrow[d] & \mathbb{C}/\mathbb{Z}(2) \arrow[r, "e^{\frac{\bullet}{2\pi i}}"] \arrow[d, "i\Im"] & \mathbb{C}^* \arrow[d, "|\bullet|"] \\
\mathcal{B}(\mathbb{C}) \arrow[r, "R"] & i\mathbb{R} \arrow[r, "e^{\frac{\bullet}{2\pi i}}"] & \mathbb{R}^+
\end{tikzcd}$$

For a proof of the commutativity of the left square, see [DZ, Prop.4.6]. Let us end this section with a remark.

Remark 3.6. Even though the regulators R and $\hat{R}$ are defined on the (extended) pre-Bloch groups, the following diagram is *not* commutative:

$$\begin{tikzcd}
\widehat{\mathcal{B}(\mathbb{C})} \arrow[r, "\hat{R}"] \arrow[d] & \mathbb{C}/(2\pi^2\mathbb{Z}) \arrow[d, "i\Im"] \\
\mathcal{B}(\mathbb{C}) \arrow[r, "R"] & i\mathbb{R}
\end{tikzcd}$$

3.3. Two cousins of the extended Bloch group: $K_3^{\mathrm{ind}}(\mathbb{C})$ and $\mathrm{CH}^2(\mathbb{C},3)$.

The Bloch group $\mathcal{B}(F)$ of a field has two cousins: $K_3^{\mathrm{ind}}(F)$ and the higher Chow groups $\mathrm{CH}^2(F,3)$, also defined by Bloch; see [B2]. A spectral sequence argument (attributed to Bloch and Bloch-Lichtenbaum) and some low-degree computations imply that for every infinite field F we have an isomorphism:

$$K_3^{\mathrm{ind}}(F) \cong \mathrm{CH}^2(F,3). \tag{32}$$

For a detailed discussion, see [E-V, Prop.5.5.20]. On the other hand, in [Su1, Thm.5.2] Suslin proves the existence of a short exact sequence:

$$0 \longrightarrow \widetilde{\mathrm{Tor}(\mu_F,\mu_F)} \longrightarrow K_3^{\mathrm{ind}}(F) \longrightarrow \mathcal{B}(F) \longrightarrow 0. \tag{33}$$

where $\widetilde{\mathrm{Tor}(\mu_F,\mu_F)}$ is a nontrivial extension of $\mathrm{Tor}(\mu_F,\mu_F)$, where μ_F is the roots of unity of F. For $F=\mathbb{C}$, the above short exact sequence becomes:

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow K_3^{\mathrm{ind}}(\mathbb{C}) \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow 0. \tag{34}$$

Moreover, Suslin proves in [Su1] that $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space.

On the other hand, in [GZ, Thm.3.12] Goette and Zickert prove that the extended Bloch group fits in a short exact sequence:

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\hat{\psi}} \widehat{\mathcal{B}(\mathbb{C})} \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow 0, \tag{35}$$

for an explicit map $\hat{\psi}$. Equations (32), (34), (35), together with the fact that $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space, imply the following.

Proposition 3.7. *There exist abstract isomorphisms:*

$$\widehat{\mathcal{B}(\mathbb{C})} \cong K_3^{\mathrm{ind}}(\mathbb{C}) \cong \mathrm{CH}^2(F,3). \tag{36}$$

In addition there are regulator maps (sometimes also known by the name of *cycle maps* or *Abel-Jacobi maps*):

$$\begin{aligned} R' : K_3^{\mathrm{ind}}(\mathbb{C}) &\longrightarrow H^1_D(\mathrm{Spec}(\mathbb{C}), \mathbb{Z}(2)) = \mathbb{C}/\mathbb{Z}(2) \\ R'' : \mathrm{CH}^2(F,3) &\longrightarrow H^1_D(\mathrm{Spec}(\mathbb{C}), \mathbb{Z}(2)) = \mathbb{C}/\mathbb{Z}(2) \end{aligned}$$

where H_D denotes Deligne cohomology. For an explicit formula for R'', see [KLM-S, Sec.5.7].

Let us end this section with a problem and a question. Recall that the extended Bloch group can be defined for any subfield F of the complex numbers, as discussed in [Ne].

Problem 3.8. Define explicit isomorphisms in (36) that commute with the regulator maps R, R' and R''.

We will come back to this problem in a forthcoming publication; [GZ].

Question 1. Is it true that for very subfield F of the complex numbers we have an isomorphism:

$$\widehat{\mathcal{B}(F)} \cong K_3^{\mathrm{ind}}(F)?$$

3.4. **q-terms and potential functions.** In this section, we will generate elements of $\widehat{\mathcal{B}(\mathbb{C})}$ from a general q-term $\mathfrak{t}$; see Theorem 2 below.

Definition 3.9. Consider the following multivalued function on $\mathbb{C}^{**}$:

$$\Phi : \hat{\mathbb{C}} \longrightarrow \mathbb{C}/\mathbb{Z}(1), \qquad \Phi(x) = \frac{1}{2\pi i}\left(\frac{\pi^2}{6} - \mathrm{Li}_2(x)\right). \tag{37}$$

Remark 3.3 shows that indeed Φ takes values in $\mathbb{C}/\mathbb{Z}(1)$. The function $x \mapsto \Phi(e^{2\pi i x})$ appears in work of Kashaev on the Volume Conjecture; see [Ks]. The relevance of the special function Φ is that it describes the growth rate of the q-factorials at complex roots of unity; see Lemma 8.1 in Section 8.1.

The next definition associates a potential function to a general q-term.

Definition 3.10. Given a general q-term $\mathfrak{t}$ as in (2), let us define its potential function $V_{\mathfrak{t}}$ by:

$$V_{\mathfrak{t}}(z) = \frac{1}{2\pi i} Q(\mathrm{Log}(z)) + \frac{1}{2\pi i}\mathrm{Log}\epsilon \cdot \mathrm{Log}(z^L) + \sum_{j=1}^{J} \epsilon_j \Phi(z^{A_j}) \tag{38}$$

where $z = (z_0, z_1, \dots, z_r)$. Let $\widehat{X_{\mathfrak{t}}}$ and denote the *critical points* of the potential $V_{\mathfrak{t}}$.

Since $V_{\mathfrak{t}}(z)$ is a multivalued function, let us explain its domain. Given a general q-term $\mathfrak{t}$ as in (2), let denote

$$H_{\mathfrak{t}} = \{z \subset (\mathbb{C}^*)^{r+1} \, | z^L \prod_{j=1}^{J}(1 - z^{A_j}) \prod_{i=0}^{r}(1 - z_i) = 0\}. \tag{39}$$

Let $\mathcal{D}_{\mathfrak{t}}$ denote the universal abelian cover of $(\mathbb{C}^*)^{r+1} \setminus H_{\mathfrak{t}}$. Observe that for every $i = 0, \dots, r$, $j = 1, \dots, J$ there are well-defined analytic maps:

$$\pi_i, \pi_{A_j}, \pi_L : (\mathbb{C}^*)^{r+1} \setminus H_{\mathfrak{t}} \to \mathbb{C}^{**} \tag{40}$$

given by $\pi_i(z) = z_i$, $\pi_{A_j}(z) = z^{A_j}$ and $\pi_L(z) = z^L$; using the notation of (11). Lifting them to the universal abelian cover, gives rise to analytic maps:

$$\hat{\pi}_i, \hat{\pi}_{A_j}, \hat{\pi}_L : \mathcal{D}_{\mathfrak{t}} \to \hat{\mathbb{C}}. \tag{41}$$

We denote the image of $z \in \mathcal{D}_{\mathfrak{t}}$ under these maps by z_i, z^{A_j} and z^L respectively. With these conventions we have:

Lemma 3.11. *Equation* (38) *defines an analytic function:*

$$V_{\mathfrak{t}} : \mathcal{D}_{\mathfrak{t}} \longrightarrow \mathbb{C}/\mathbb{Z}(1). \tag{42}$$

The next proposition describes the critical points and the critical values of the potential function.

Proposition 3.12. (a) The critical points z of $V_{\mathfrak{t}}$ are the solutions to the following system of *Logarithmic Variational Equations*:

$$\sum_{j=1}^{J} \epsilon_j v_i(A_j) \mathrm{Log}(1 - z^{A_j}) + \frac{\partial Q}{\partial z_i}(\mathrm{Log}(z)) + \mathrm{Log}\epsilon \cdot v_i(L) = 0 \tag{43}$$

for $i = 0, \dots, r$.
(b) There is a map:

$$\widehat{X_{\mathfrak{t}}} \longrightarrow X_{\mathfrak{t}}, \qquad z \mapsto (\pi(z_1), \dots, \pi(z_r)) \tag{44}$$

where $\pi : \hat{\mathbb{C}} \to \mathbb{C}^{**}$ is the projection map.

For $A = A_j$, $(j = 1, \dots, J)$ or $A = L$, let $p_{z,A} \in 2\mathbb{Z}$ (or simply, p_A if z is clear) be defined so that we have:

$$\mathrm{Log}(z^A) = \sum_{i=0}^{r} v_i(A) \mathrm{Log}(z_i) - p_{z,A}\pi i. \tag{45}$$

Given $w = (x; p_0, q_0) \in \hat{\mathbb{C}}$ and even integers $p, q \in 2\mathbb{Z}$ let us denote

$$T_1^p T_0^q(w) := (x; p_0 + p, q_0 + q) \in \hat{\mathbb{C}} \tag{46}$$

In other words, T_1 and T_0 are generators for the deck transformations of the $\mathbb{Z}^2$-cover:

$$\pi : \hat{\mathbb{C}} \longrightarrow \mathbb{C}^{**}. \tag{47}$$

We need one more piece of notation: for $z \in \mathcal{D}_{\mathfrak{t}}$, consider the corresponding elements z^L and z_i for $i = 0, \dots, r$ $\hat{\mathbb{C}}$. We denote by $z^{-L/2} \in \hat{\mathbb{C}}$ the unique element of $\hat{\mathbb{C}}$ that solves the equation:

$$-\mathrm{Log}(z^L) - \mathrm{Log}(z^{-L/2}) + \frac{1}{2} \sum_{i=0}^{r} v_i(L) \mathrm{Log}(z_i) = 0. \tag{48}$$

Definition 3.13. With the above conventions, consider the map

$$\hat{\beta}_{\mathfrak{t}} : \widehat{X_{\mathfrak{t}}} \longrightarrow \widehat{\mathcal{P}(\mathbb{C})} \tag{49}$$

given by:

$$w \mapsto \hat{\beta}_{\mathfrak{t}}(w) := [z^{-L/2}; 0, 2\frac{\mathrm{Log}\epsilon}{\pi i}] - [z^{-L/2}; 0, 0] + \sum_{j=1}^{J} \epsilon_j [T_1^{p_{z,A_j}}(z^{A_j})] \tag{50}$$

Our next definition assigns an important numerical invariant to a special term $\mathfrak{t}$.

Definition 3.14. For a general q-term $\mathfrak{t}$, let

$$\mathrm{CV}_{\mathfrak{t}} = \{e^{-V_{\mathfrak{t}}(z)} \,|z \text{ satisfies } (43)\}. \tag{51}$$

Our next main theorem links the exponential of the critical values of $V_{\mathfrak{t}}$ with the values of $\hat{\beta}_{\mathfrak{t}}$ under the regulator map. Let $\mathcal{P}$ denote the countable set of *periods* in the sense of Kontsevich-Zagier, [KZ].

Theorem 2. (a) The map $\hat{\beta}_{\mathfrak{t}}$ descends to a map

$$\hat{\beta}_{\mathfrak{t}} : \widehat{X_{\mathfrak{t}}} \longrightarrow \widehat{\mathcal{B}(\mathbb{C})} \tag{52}$$

which we denote by the same name.
(b) We have a commutative diagram:

$$\begin{tikzcd} \widehat{X_{\mathfrak{t}}} \arrow[rr, "\hat{\beta}_{\mathfrak{t}}"] \arrow[dr, "e^{-V_{\mathfrak{t}}}"'] & & \widehat{\mathcal{B}(\mathbb{C})} \arrow[dl, "e^{\frac{1}{2\pi i}\hat{R}}"] \\ & \mathbb{C}^* & \end{tikzcd}$$

(c) $\mathrm{CV}_{\mathfrak{t}}$ is a finite subset of $e^{\mathcal{P}}$. Thus, we get a map:

$$\mathrm{CV} : \text{General } q\text{-terms} \longrightarrow \text{Finite subsets of } e^{\mathcal{P}}. \tag{53}$$

Example 3.15. Let us continue with the Example 1.5. When $a = -1$, $b = 2$ and $\epsilon = -1$, the Variational Equations (17) have solutions

$$\{z^{\pm 1} \,|\, z = e^{2\pi i/6}\}.$$

In that case, the corresponding elements of the extended Bloch group are

$$2[z^{\pm 1}; 0, 0] + [z^{-1/2}; 0, 2] - [z^{-1/2}; 0, 0],$$

and their values under the regulator map $\hat{R}$ are given by:

$$0 \pm i\, 2.0298832128193074\dots$$

whose imaginary part equals to the *Volume* $\mathrm{Vol}(4_1)$ of the 4_1 knot; see [Th]. Moreover,

$$\begin{aligned} \mathrm{CV}_{\mathfrak{t}} &= \{e^{\pm \frac{1}{2\pi}\mathrm{Vol}(4_1)}\} \\ &= \{0.7239261119\dots, 1.3813564445\dots\}. \end{aligned}$$

4. Proofs

In this section we will give the proofs of Proposition 3.12 and Theorem 2.

4.1. **Proof of Proposition 3.12.** It suffices to show that for every $i = 0, \dots, r$, $2\pi\sqrt{-1} z_i \frac{\partial V_{\mathfrak{t}}}{\partial z_i}$ is given by the left hand side of Equation (43). This follows easily from the definition of $V_{\mathfrak{t}}$ and the elementary computation:

$$\Phi'(x) = \frac{1}{2\pi i} \frac{\mathrm{Log}(1-x)}{x}. \tag{54}$$

4.2. **Proof of Theorem 2.** The proof is similar to the proof of Theorem 1, once we keep track of the branches of the logarithms.

Fix a general q-term $\mathfrak{t}$ and consider the map given by (50). Let us fix $z = (z_0, \dots, z_r)$ that satisfies the Variational Equations (12). We will show that $\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z)) = 0 \in \mathbb{C} \wedge \mathbb{C}$, where $\hat{\nu}$ is given by (22). We have

$$\hat{\beta}_{\mathfrak{t}}(z) = \hat{\beta}_{1,\mathfrak{t}}(z) + \hat{\beta}_{2,\mathfrak{t}}(z)$$

where

$$\begin{aligned} \hat{\beta}_{1,\mathfrak{t}}(z) &= \sum_{j=1}^{J} \epsilon_j [T_1^{p_{z,A_j}}(z^{A_j})] \\ \hat{\beta}_{2,\mathfrak{t}}(w) &= [z^{-L/2}; 0, 2\frac{\mathrm{Log}\epsilon}{\pi i}] - [z^{-L/2}; 0, 0]. \end{aligned}$$

It follows that

$$\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z)) = \hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z)) + \hat{\nu}(\hat{\beta}_{2,\mathfrak{t}}(z))$$

where

$$\begin{aligned} \hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z)) &= \sum_{j=1}^{J} \epsilon_j (\mathrm{Log}(z^{A_j}) + p_{z,A_j}\pi i) \wedge (-\mathrm{Log}(1 - z^{A_j})) \\ \hat{\nu}(\hat{\beta}_{2,\mathfrak{t}}(z)) &= \mathrm{Log}(z^{-L/2}) \wedge 2\mathrm{Log}\epsilon \\ &= 2\mathrm{Log}(z^{-L/2}) \wedge \mathrm{Log}\epsilon. \end{aligned}$$

On the other hand, using Equation (45), we have:

$$\begin{aligned} (\mathrm{Log}(z^A) + p_{z,A}\pi i) \wedge \mathrm{Log}(1 - z^A) &= \sum_{i=0}^{r} v_i(A)(\mathrm{Log}(z_i) \wedge \mathrm{Log}(1 - z^A)) \\ &= \sum_{i=0}^{r} \mathrm{Log}(z_i) \wedge (v_i(A)\mathrm{Log}(1 - z^A)). \end{aligned}$$

Since z satisfies the Logarithmic Variational Equations (43), after we interchange the j and i summation, we obtain that:

$$\begin{aligned} \hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z)) &= \sum_{i=0}^{r} \sum_{j=1}^{J} \mathrm{Log}(z_i) \wedge (-\epsilon_j v_i(A_j)\mathrm{Log}(1 - z^{A_j})) \\ &= \sum_{i=0}^{r} \mathrm{Log}(z_i) \wedge (\sum_{j=1}^{J} -\epsilon_j v_i(A_j)\mathrm{Log}(1 - z^{A_j})) \\ &= \sum_{i=0}^{r} \mathrm{Log}(z_i) \wedge (\frac{\partial Q}{\partial z_i}(\mathrm{Log}(z)) + \mathrm{Log}\epsilon \cdot v_i(L)). \end{aligned}$$

Since Q is an integral symmetric bilinear form and $\wedge$ is skew-symmetric, it follows that

$$\sum_{i=0}^{r} \mathrm{Log}(z_i) \wedge (\frac{\partial Q}{\partial z_i}(\mathrm{Log}(z)) = 0.$$

Moreover,

$$\begin{aligned}\sum_{i=0}^{r}\mathrm{Log}(z_i)\wedge(\mathrm{Log}\epsilon\cdot v_i(L)) &= \sum_{i=0}^{r}v_i(L)\mathrm{Log}(z_i)\wedge\mathrm{Log}\epsilon\\ &= (\mathrm{Log}(z^L)+p_{z,L}\pi i)\wedge\mathrm{Log}\epsilon\\ &= \mathrm{Log}(z^L)\wedge\mathrm{Log}\epsilon.\end{aligned}$$

Thus,

$$\hat{\nu}(\hat{\beta}_{1,\mathfrak{t}}(z))=\mathrm{Log}(z^L)\wedge\mathrm{Log}\epsilon$$

which implies that

$$\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z))=(2\mathrm{Log}(z^{-L/2})+\mathrm{Log}(z^L))\wedge\mathrm{Log}\epsilon.$$

On the other hand, reducing Equation (48) modulo $\pi i\mathbb{Z}$, and using (45) we have:

$$\begin{aligned}0 &= -\mathrm{Log}(z^L)-\mathrm{Log}(z^{-L/2})+\frac{1}{2}\sum_{i=0}^{r}v_i(L)\mathrm{Log}(z_i)\\ &= -\mathrm{Log}(z^L)-\mathrm{Log}(z^{-L/2})+\frac{1}{2}(\mathrm{Log}(z^L)+\pi i p_{z,L})\\ &= -\frac{1}{2}(\mathrm{Log}(z^L)+2\mathrm{Log}(z^{-L/2}))+\frac{\pi i p_{z,L}}{2}\\ &= -\frac{1}{2}(\mathrm{Log}(z^L)+2\mathrm{Log}(z^{-L/2})).\end{aligned}$$

In other words, we have:

$$\mathrm{Log}(z^L)+2\mathrm{Log}(z^{-L/2})\in\frac{\pi i}{2}\mathbb{Z}. \tag{55}$$

Since $\mathrm{Log}\epsilon\in\pi i\mathbb{Z}$, it follows that $\hat{\nu}(\hat{\beta}_{\mathfrak{t}}(z))=0$ and concludes part (a) of Theorem 2.

For part (b), observe first that for every $w\in\hat{\mathbb{C}}$ and every even integers p and q we have:

$$\hat{R}([T_1^pT_0^q(w)])=\hat{R}([w])+\frac{\pi i}{2}(q\mathrm{Log}(z)+p\mathrm{Log}(1-z)).$$

Moreover, by the definition of Φ and by Equation (45), for any integral linear form $A=A_j$ for $j=1,\dots,J$ or $A=L$, we have:

$$-\Phi(z^A) \;=\; \frac{1}{2\pi i}\left(\hat{R}([T_1^{p_{z,A}}(z^A)])-\frac{1}{2}\mathrm{Log}(z^A)\mathrm{Log}(1-z^A)-\frac{\pi i}{2}p_{z,A}\mathrm{Log}(1-z^A)\right).$$

Thus, using the definition of the potential function from Equation (38), we obtain that:

$$-V_{\mathfrak{t}}(z)=\frac{1}{2\pi i}R(\hat{\beta}_{\mathfrak{t}}(z))+T_1+T_2$$

where

$$\begin{aligned}T_1 &= \frac{1}{2\pi i}\left(-\frac{1}{2}\sum_j\epsilon_j\mathrm{Log}(z^{A_j})\mathrm{Log}(1-z^{A_j})-\frac{\pi i}{2}p_{z,A_j}\mathrm{Log}(1-z^{A_j})\right)\\ T_2 &= -\frac{1}{2\pi i}Q(\mathrm{Log}(z))-\frac{1}{2\pi i}\mathrm{Log}\epsilon\cdot\mathrm{Log}(z^L)-\frac{1}{2\pi i}(\hat{R}([z^{-L/2};0,\frac{2\mathrm{Log}\epsilon}{\pi i}]-[z^{-L/2};0,0])\\ &= -\frac{1}{2\pi i}Q(\mathrm{Log}(z))-\frac{1}{2\pi i}\mathrm{Log}\epsilon\cdot\mathrm{Log}(z^L)-\frac{\mathrm{Log}\epsilon}{2\pi i}\mathrm{Log}(z^{-L/2}).\end{aligned}$$

Using (45), and interchanging i and j summation, and using the Logarithmic Variational Equations (43), it follows that:

$$\begin{aligned}
\sum_{j=0}^{J} \epsilon_j \mathrm{Log}(z^{A_j}) \mathrm{Log}(1-z^{A_j}) &= \sum_{j=0}^{J} (\sum_{i=0}^{r} \epsilon_j v_i(A_j) \mathrm{Log}(z_i) - p_{z,A_j} \pi i) \mathrm{Log}(1-z^{A_j}) \\
&= \sum_{i=0}^{r} \mathrm{Log}(z_i) \sum_{j=0}^{J} \epsilon_j v_i(A_j) \mathrm{Log}(1-z^{A_j}) - \sum_{j=0}^{J} \epsilon_j p_{z,A_j} \pi i \mathrm{Log}(1-z^{A_j}) \\
&= \sum_{i=0}^{r} \mathrm{Log}(z_i)(-\frac{\partial Q}{\partial z_i} - \mathrm{Log}\epsilon \cdot v_i(L)) - \sum_{j=0}^{J} \epsilon_j p_{z,A_j} \pi i \mathrm{Log}(1-z^{A_j}).
\end{aligned}$$

Thus,

$$T_1 = \frac{1}{4\pi i} \sum_{i=0}^{r} \mathrm{Log}(z_i) \frac{\partial Q}{\partial z_i} (\mathrm{Log}(z)) + \frac{\mathrm{Log}\epsilon}{4\pi i} \sum_{i=0}^{r} v_i(L) \mathrm{Log}(z_i).$$

Since Q is a symmetric bilinear form, it follows that

$$\frac{1}{2} \sum_{i=0}^{r} \mathrm{Log}(z_i) \frac{\partial Q}{\partial z_i} (\mathrm{Log}(z)) - Q(\mathrm{Log}(z)) = 0.$$

This, together with Equation (48) implies that $T_1 + T_2 = 0$. Exponentiating, it follows that

$$e^{-V_{\mathfrak{t}}(z)} = e^{\frac{1}{2\pi i} \hat{R}(\hat{\beta}_{\mathfrak{t}}(z))} \in \mathbb{C}^*$$

which concludes part (b) of Theorem 2.

For part (c), since an analytic function is constant on a connected component of its sets of critical points, and since the set of critical points $\widehat{X_{\mathfrak{t}}}$ is the set of complex points of an affine variety defined over $\mathbb{Q}$, it follows that $\widehat{X_{\mathfrak{t}}}$ has finitely many connected components. Thus, $\mathrm{CV}_{\mathfrak{t}}$ is a finite subset of $\mathbb{C}$. Moreover, since the value of the regulator at a point $[z; p, q]$ with $z \in \overline{\mathbb{Q}}$ is a period, in the sense of Kontsevich-Zagier, and since every connected component of $\widehat{X_{\mathfrak{t}}}$ has a point w so that $z = e^{2\pi i w} \in (\overline{\mathbb{Q}})^{r+1}$ (due to the Variational Equations (12)), it follows that $\mathrm{CV}_{\mathfrak{t}}$ is a subset of $e^{\mathcal{P}}$. This concludes the proof of Theorem 2. ☐

5. SPECIAL q-TERMS, GENERATING SERIES AND SINGULARITIES

In this section, which is independent from the previous ones, we will assign two germs of analytic functions $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ to a special q-term, and we will formulate a conjecture regarding their analytic continuation in the complex plane minus a set of singularities related to the image of the composition $\hat{R} \circ \hat{\beta}_{\mathfrak{t}}$.

5.1. **From special q-terms to elements of the Habiro ring.**

In the first part of the paper, we assigned elements of the extended Bloch group to a general q-term $\mathfrak{t}$, and used them to construct a finite set $\mathrm{CV}_{\mathfrak{t}}$ of $e^{\mathcal{P}}$.

In the second part, which is largely independent from the first, we will assign two series $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ to a special q-term $\mathfrak{t}$ and make a conjecture regarding the set of singularities $S_{\mathfrak{t}}$ of the analytic continuation of $L^{\mathrm{np}}_{\mathfrak{t}}(z)$.

Before we give details, we should point out that *integrality* (in the sense of Laurent polynomials in $\mathbb{Z}[q^{\pm 1}]$) plays an important role in this section. Precisely for this reason, we consider the special q-terms of the form (1) and *not* the general q-terms of the form (2).

The next definition assigns a sequence of Laurent polynomials to a special q-term.

Definition 5.1. Given a special q-term $\mathfrak{t}_{n,k}(q)$ as in (1), consider the sequence $(a_{\mathfrak{t},n}(q))$ of Laurent polynomials:

$$a_{\mathfrak{t},n}(q) = \sum_{k=0}^{\infty} \mathfrak{t}_{n,k}(q) \in \mathbb{Z}[q^{\pm 1}] \tag{56}$$

(the sum is finite for every $n \in \mathbb{N}$) and the corresponding expression:

$$G_{\mathfrak{t}}(q) = \sum_{n=0}^{\infty} (q)_n a_{\mathfrak{t},n}(q). \tag{57}$$

The above expression is indeed infinite, and does not make sense as an analytic function inside or outside the unit disk. Because of the innocent-looking $(q)_n$, the above expression makes sense in the *Habiro ring*. The latter was introduced by Habiro in [Ha2]:

$$\widehat{\Lambda} := \lim_{\substack{\longleftarrow \\ n}} \mathbb{Z}[q]/((1-q)(1-q^2)\dots(1-q^n)). \tag{58}$$

As a set, $\widehat{\Lambda}$ contains all series of the form

$$f(q) = \sum_{n=0}^{\infty} f_n(q)\,(1-q)(1-q^2)\dots(1-q^n), \qquad \text{where} \quad f_n(q) \in \mathbb{Z}[q^{\pm 1}], \tag{59}$$

with the warning that $f(q)$ does *not* uniquely determine $(f_n(q))$.

5.2. **Two generating power series from an element of the Habiro ring.** Fix a special q-term $\mathfrak{t}$. The element $G_{\mathfrak{t}}(q)$ of the Habiro ring gives rise to two generating series:

$$L^{\mathrm{np}}_{\mathfrak{t}}(z) = \sum_{n=1}^{\infty} G_{\mathfrak{t}}(e^{\frac{2\pi i}{n}}) z^n \tag{60}$$

$$L^{\mathrm{p}}_{\mathfrak{t}}(z) = \mathcal{B}(G_{\mathfrak{t}}(e^{1/x})) \tag{61}$$

where $\mathcal{B}$ is the *Borel transform* defined by

$$\mathcal{B} : \mathbb{C}[[\frac{1}{x}]] \longrightarrow \mathbb{C}[[z]], \qquad \mathcal{B}\left(\frac{1}{x^{n+1}}\right) = \frac{z^n}{n!} \tag{62}$$

Remark 5.2. In [Ha2] Habiro proves each of the following series $G_{\mathfrak{t}}(q)$, $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ or $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ uniquely determines all others. However, note that $G_{\mathfrak{t}}(q)$ does *not* determine $\mathfrak{t}$.

The next lemmas, that use standard estimates from [GL2] and [GL3, Thm.6], show that $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ are analytic functions at $z = 0$.

Lemma 5.3. *For every special q-term $\mathfrak{t}$, $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ are analytic at $z = 0$.*

Proof. We need to recall what is nicely bounded sequence, from [GL3, Def.1.3]: given a sequence $(f_n(q))$ of Laurent polynomials $f_n(q) \in \mathbb{Z}[q^{\pm 1}]$ for $n \in \mathbb{N}$, we say that $(f_n(q))$ is *nicely bounded* if there exist the bounds on their span and coefficients. That is, there are constants $C, C' > 0$ that depend on $(f_n(q))$ such that for $n > 0$,

$$\mathrm{span}_q f_n(q) \subset [-C' n^2, C' n^2] \tag{63}$$

$$||f_n(q)||_1 \leq C^n. \tag{64}$$

Here, the *span* and the *ℓ^1-norm* of a Laurent polynomial $f(q) = \sum_{j=m}^{M} c_j q^j$ with $a_m a_M \neq 0$ are defined to be $[-m, M]$ and

$$||f(q)||_1 = \sum_{j=m}^{M} |c_j|, \tag{65}$$

respectively. Given a sequence $(f_n(q))$ of Laurent polynomials, let

$$G_f(q) = \sum_{n=0}^{\infty} (q)_n a_{\mathfrak{t},n}(q) \tag{66}$$

denote the corresponding element of the Habiro ring, and let us consider the power series

$$L_f^{\mathrm{np}}(z) = \sum_{n=1}^{\infty} G_f(e^{\frac{2\pi i}{n}}) z^n \tag{67}$$

$$L_f^{\mathrm{p}}(z) = \mathcal{B}(G_f(e^{1/x})) \tag{68}$$

In [GL3, Thm.6] T.T.Q. Le and the author showed that if $(f_n(q))$ is nicely bounded, then $G_f(z)$ is Gevrey-1. Thus $L_f^{\mathrm{p}}(z)$ is convergent for $z = 0$.

Moreover, it is easy to see that if $(f_n(q))$ is nicely bounded, then there exists a constant $C > 0$ that depends on f so that for every $N > 0$ we have:

$$|G_f(e^{\frac{2\pi i}{N}})| \leq C^N. \tag{69}$$

Indeed, since $|e^{\frac{2\pi i}{N}}| = 1$, we have:

$$\begin{aligned} |G_f(e^{\frac{2\pi i}{N}})| &= |\sum_{n=0}^{N} (e^{\frac{2\pi i}{N}})_n f_n(e^{\frac{2\pi i}{N}})| \\ &\leq \sum_{n=0}^{N} ||(q)_n f_n(q)|||_1 \\ &\leq \sum_{n=0}^{N} C'^n \\ &\leq C^N. \end{aligned}$$

Thus, to finish the proof of the Lemma, it suffices to show that for every special q-term $\mathfrak{t}$, the sequence $(a_{\mathfrak{t},n}(q))$ of Laurent polynomials is nicely bounded.

To show this, observe that there exist a constant $C_1 > 0$ so that if $f_{n,k}(q) \neq 0$, then $k = (k_1, \dots, k_r)$ where $0 \geq k_i < Cn$ for all $i = 1, \dots, r$. In other words, the summation in (56) involves only polynomially many nonzero terms in n.

Since a q-binomial is a Laurent polynomial with nonnegative coefficients, its ℓ^1 norm equals to the corresponding binomial coefficient. This, and the above, implies that there exists $C_2 > 0$ so that

$$||t_{n,k}(q)||_1 \leq C_2^n$$

for all (n, k). Since the summation in (56) involves only polynomially many nonzero terms in n, it follows that there exists $C_3 > 0$ so that

$$||a_{\mathfrak{t},n}(q)||_1 \leq C_3^n$$

for all $n > 0$. Moreover, for k linearly bounded by n, it is easy to see that there exist a constant $C_4 > 0$ so that for all (n, k) with $n > 0$ we have:

$$\mathrm{span}\mathfrak{t}_{n,k}(q) \subset [-C_4\, n^2, C_4\, n^2]$$

Consequently, there exists a constant $C_5 > 0$ so that for all $n > 0$ we have:

$$\mathrm{span} a_{\mathfrak{t},n}(q) \subset [-C_5\, n^2, C_5\, n^2].$$

This concludes the proof of the fact that $(a_{\mathfrak{t},n}(q))$ is nicely bounded, and the proof of Lemma 5.3. □

5.3. **The Newton polytope of a q-term.** Let us recall the Newton polytope of a q-term.

Definition 5.4. Given a q-term $\mathfrak{t}$ as in (1), define its *Newton polytope* $P_{\mathfrak{t}}$ by:

$$P_{\mathfrak{t}} = \{w \in \mathbb{R}^r \mid B_j(w) \geq 0,\ C_j(w) \geq 0,\ D_j(w) \geq 0 \,\text{for}\, j = 1, \dots, J\} \subset \mathbb{R}^r. \tag{70}$$

By our assumptions on the q-term $\mathfrak{t}$, $P_{\mathfrak{t}}$ is a compact rational convex polytope in $\mathbb{R}^r$ with non-empty interior. Moreover, for every $n \in \mathbb{N}$ we have:

$$\text{support}(\mathfrak{t}_{n,k}) = nP_{\mathfrak{t}} \cap \mathbb{Z}^r. \tag{71}$$

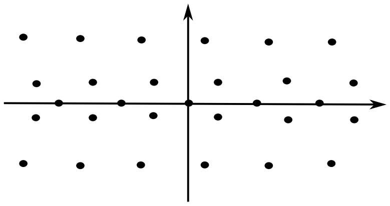
It is easy to see that the restriction $\mathfrak{t}|_\Delta$ of a q-term $\mathfrak{t}$ to a face Δ of its Newton polytope is also a q-term.

5.4. **An ansatz for the singularities of $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{p}}_{\mathfrak{t}}(z)$.** Recall the finite set $\mathrm{CV}_{\mathfrak{t}} \subset \mathbb{C}^*$ from Definition 3.14 and Theorem 2 associated to a q-term.

Definition 5.5. For a q-term $\mathfrak{t}$, define

$$S_{\mathfrak{t}} := \{0\} \cup \cup_{\Delta\ \text{face of}\ P_{\mathfrak{t}}} \mathrm{CV}_{\mathfrak{t}|_\Delta} \tag{72}$$

It follows that $2\pi i \log(S_{\mathfrak{t}}) + \mathbb{Z}(2) \subset \mathbb{C}$ is a discrete subset of a finite union of horizontal lines in the complex plane and may be depicted as follows:



Here, the horizontal spacing between two dots in any horizontal line is $4\pi^2 = 39.4784176044\dots$.

Our ansatz relates the singularities of the analytic continuation of the germs $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ with the sets $S_{\mathfrak{t}}$ and $2\pi i \log(S_{\mathfrak{t}}) + \mathbb{Z}(2)$.

Keeping in mind the extra term $(q)_n$ in Equation (57), given a special q-term $\mathfrak{t}_{n,k}$ let us denote the special q-term $\tilde{\mathfrak{t}}_{n,k}$ by:

$$\tilde{\mathfrak{t}}_{n,k}(q) := (q)_n \mathfrak{t}_{n,k}(q). \tag{73}$$

Conjecture 1. *For every special q-term $\mathfrak{t}$, the germs $L^{\mathrm{np}}_{\mathfrak{t}}(z)$ and $L^{\mathrm{p}}_{\mathfrak{t}}(z)$ has an analytic continuation as a multivalued function in $\mathbb{C} \setminus S_{\tilde{\mathfrak{t}}}$ and $\mathbb{C} \setminus 2\pi i \log(S_{\tilde{\mathfrak{t}}}) + \mathbb{Z}(2)$, respectively. Moreover, at a singular point λ, each germ has a local expansion of the form:*

$$(z-\lambda)^{\alpha_\lambda} (\log(z-\lambda))^{\beta_\lambda} h_\lambda(z-\lambda)$$

where $\alpha_\lambda \in (1/2)\mathbb{Z}$, $\beta_\lambda \in \mathbb{N}$ and $h_\lambda(w)$ is an analytic function at $w = 0$.

In a subsequent publication we will show how Conjecture 1 implies a strong form of the Volume Conjecture, with exponentially small terms included; see [GL2].

Example 5.6. If the special q-term is given by (16), then Conjecture 1 is known; see [CG]. There is ample numerical and theoretical experimentation when $a = -1, b = 2\ \epsilon = -1$ in (16); see [Sh].

6. SPECIAL q-TERMS AND 3-DIMENSIONAL KNOTTED OBJECTS

6.1. **A source of special q-terms: 3-dimensional knotted objects.** The remaining sections are the third part of the paper. Our aim is to give examples of special q-terms of interest to 3-dimensional topology, to motivate our choice of potential functions, and to compare with other combinatorial encodings of general q-terms, due to Kontsevich and Neumann-Zagier.

To begin with, general q-terms are easy to produce: see Remark 1.2. A rich source of special q-terms is also 3-dimensional quantum topology which has been a motivation for much of our work. This is already observed in [GL1] (much in the spirit of [Z] and [WZ]), where T.T.Q. Le and the author assign a special q-term $\mathfrak{t}$ to a planar projection of a knot, and show that the corresponding sequence of Laurent polynomials $(a_{\mathfrak{t},n}(q))$ is q-holonomic. See [GL1, Thm.1] and [GL1, Sec.3.2].

Let us explain what is a knotted object.

Definition 6.1. A *knotted object* will either denote a closed 3-manifold which is an integer homology sphere, or the complement of a knot in S^3.

In this section, K will denote a knot, M will denote an integer homology 3-sphere, and $\mathcal{K}$ will denote a knotted object.

A refined version of the state-sum formulas for the colored Jones polynomial of [GL1, Sec.3.2] which takes into account the non-commutative version of the MacMahon Master Theorem (see [GLZ]) as well as the work of [HL], imply the following result.

Let

$$Z^{\kappa} : \text{Knots} \longrightarrow \widehat{\Lambda} \tag{74}$$

denote the lifted Kashaev invariant of a knot, as in [HL]. In [GL4] we will show the following.

Theorem 3. [GL4] *There exists a map:*

$$\textit{Planar Diagrams of long Knots} \longrightarrow \textit{Special q-terms} \tag{75}$$

that fits in a commutative diagram:

$$\begin{tikzcd}
\textit{Planar Diagrams of long Knots} \arrow[r] \arrow[d, "closure"'] & \textit{Special q-terms} \arrow[d, "(G_\bullet(q))"] \\
\textit{Knots} \arrow[r, "Z^{\kappa}"] & \widehat{\Lambda}.
\end{tikzcd}$$

If PD is a planar projection of a long knot and $\mathfrak{t}_{n,k}$ is the corresponding special q-term from (75), where $k = (k_1, \dots, k_r)$, then I is the number of crossings of PD.

One of Habiro's motivations for introducing his ring, was to construct an invariant of integer homology spheres that encodes the $\mathfrak{sl}_2$ quantum invariants of Reshetikhin-Turaev; see [Ha1] and [Tu]. Habiro's construction was extended to simple Lie algebras by Habiro and the second author (see [HL]):

$$Z^{HL} : \text{Integer Homology Spheres} \times \text{Simple Lie algebras} \longrightarrow \widehat{\Lambda} \tag{76}$$

Recall that every integer homology sphere can be obtained by surgery on a *unit-framed algebraically split* link in S^3; i.e., a link whose linking matrix is diagonal with ±1 in the diagonal. Let us abbreviate those links by AS-links below.

Conjecture 2. *There exists a map:*

$$\textit{Planar Diagrams of AS-links} \longrightarrow \textit{Special q-terms} \tag{77}$$

that fits in a commutative diagram:

$$\begin{tikzcd}
\textit{Planar Diagrams of long Knots} \arrow[r] \arrow[d, "surgery"'] & \textit{Special q-terms} \arrow[d, "(G_\bullet(q))"] \\
\textit{Integer Homology Spheres} \arrow[r, "Z^{HL}"] & \widehat{\Lambda}.
\end{tikzcd}$$

6.2. **The $3 = 3$ principle.** In this section we will describe a classical topological construction that assigns elements of K-theory to knotted 3-dimensional objects. This section is slightly more abstract and independent from the rest of the results. The main idea is a numerical coincidence between the dimension of the knotted objects with the index of $K_3(\mathbb{C})$.

Recall that the fundamental group $\pi_1(S^3 - K)$ of a knot complement has a *peripheral subgroup* generated by the inclusion of the boundary torus $\partial(S^3 - K)$ in $S^3 - K$. An $\mathrm{SL}(N, \mathbb{C})$-*representation* of a group G is a homomorphism $\rho : G \longrightarrow \mathrm{SL}(N, \mathbb{C})$. A *parabolic*-$\mathrm{SL}(N, \mathbb{C})$ representation of the complement of a knot K is a representation

$$\rho : \pi_1(S^3 - K) \longrightarrow \mathrm{SL}(N, \mathbb{C})$$

whose restriction to the boundary torus maps to a parabolic subgroup of $\mathrm{SL}(N, \mathbb{C})$; i.e., a subgroup conjugate to the upper triangular matrices with 1 on the diagonal. Below, the superscript *par* will denote parabolic representations.

Definition 6.2. For a natural number $N \geq 2$, and an integer homology sphere M, we define the moduli space of $\mathrm{SL}(N, \mathbb{C})$ representations as follows:

$$X_{M,N} = \mathrm{Hom}(\pi_1(M), \mathrm{SL}(N, \mathbb{C}))/\mathrm{SL}(N, \mathbb{C}) \tag{78}$$

For a knot K, we define the moduli space of $\mathrm{SL}(N, \mathbb{C})$ parabolic representations as follows:

$$X_{K,N} = \mathrm{Hom}^{\mathrm{par}}(\pi_1(M), \mathrm{SL}(N, \mathbb{C}))/\mathrm{SL}(N, \mathbb{C}) \tag{79}$$

It is well-known that for every knotted object $\mathcal{K}$, the moduli space $X_{\mathcal{K},N}$ is the set of complex points of an affine variety defined over $\mathbb{Q}$.

Given a parabolic $\mathrm{SL}(N, \mathbb{C})$-representation ρ of a knotted object, consider the corresponding classifying space map $[M, B\mathrm{SL}(N, \mathbb{C})]$, as well as the composite map:

$$[M, B\mathrm{SL}(N, \mathbb{C})] \to [M, B\mathrm{SL}(N, \mathbb{C})^\delta] \to [M, B\mathrm{GL}(N, \mathbb{C})^\delta] \to [M, B\mathrm{GL}(N, \mathbb{C})^{\delta,+}]. \tag{80}$$

where $\mathrm{SL}(N, \mathbb{C})^\delta$ denotes the group $\mathrm{SL}(N, \mathbb{C})$ with the discrete topology, and the superscript $+$ denotes Quillen's plus construction. This gives rise to an element of $K_3(\mathbb{C})$, and after projection, also gives an element of $K_3^{\mathrm{ind}}(\mathbb{C})$. Altogether, we obtain the following.

Proposition 6.3. *For every knotted object $\mathcal{K}$, we have a map:*

$$\beta_{\mathcal{K},N} : X_{\mathcal{K},N} \longrightarrow K_3^{\mathrm{ind}}(\mathbb{C}) \tag{81}$$

One expects that if M is a homology sphere, and $\mathfrak{t}$ is a term associated to the $\mathrm{SL}(N, \mathbb{C})$-quantum invariants of M via the map (77), then the image of $\beta_{M,N}$ in $K_3^{\mathrm{ind}}(\mathbb{C})$ will coincide with the image of $\beta_\mathfrak{t}$ in $\widehat{\mathcal{B}(\mathbb{C})}$ via a suitable isomorphism map of (36).

6.3. **Geometric realization of elements of the extended Bloch group.** It is natural to ask which elements of the extended Bloch group $\widehat{\mathcal{B}(\mathbb{C})}$ are obtained from the map $\hat{\beta}_\mathfrak{t}$ of (49). Let us give a heuristic argument which indicates that perhaps all elements arise this way.

Fix $x \in \widehat{\mathcal{B}(\mathbb{C})}$. Using the abstract isomorphism (36), consider a corresponding element in $K_3^{\mathrm{ind}}(\mathbb{C})$, and lift it to an element of $K_3(\mathbb{C}) = \pi_3(B\mathrm{GL}(\mathbb{C})^+)$. Since homotopy and bordism coincide in small degrees, we may represent the above element by a map $M^3 \to B\mathrm{SL}(N, \mathbb{C})$ for some closed 3-manifold M and natural number $N \in \mathbb{N}$. Now, consider the corresponding point y in the $\mathrm{SL}(N, \mathbb{C})$ character variety $\mathcal{X}_{M,N}$ of M.

If the expectation formulated in the last paragraph of Section 6.2 is true, then y lies in the image of $\beta_\mathfrak{t}$ for some q-term $\mathfrak{t}$ which gives rise to the $\mathrm{SL}(N, \mathbb{C})$ quantum invariants of M.

7. Other combinatorial encodings of q-terms

We mentioned already in Remark 1.2 that a general q-term may be encoded by a matrix of integer entries, that keeps track of the coefficients of the quadratic and linear forms and signs that appear in the very definition of a q-term. In this section we will compare general q-terms with Neumann-Zagier matrices and Kontsevich's rational polytopes.

7.1. Special q-terms are general.

In a problem list submitted to the Aarhus University in 2006, Kontsevich formulated a question regarding the growth rate of multisums of quantum factorials; see [Ko]. In this section we will show that our Conjecture 1 is a refinement of Kontsevich's question.

Lemma 7.1. *Every special q-term is a general q-term.*

Proof. We may write a special q-term $\mathfrak{t}$ of (1) in the form:

$$\mathfrak{t}_{n,k}(q) = q^{Q(n,k)}\epsilon^{L(n,k)} \prod_{j=1}^{4J} (q)_{A_j(n,k)}^{\epsilon_j}.$$

where the linear forms A_j and the signs $\epsilon_j = \pm 1$ are given by:

$$A_j = \begin{cases} B_j & \text{if } 1 \leq j \leq J \\ D_j & \text{if } J+1 \leq j \leq 2J \\ C_j & \text{if } 2J+1 \leq j \leq 3J \\ B_j - C_j & \text{if } 3J+1 \leq j \leq 4J \end{cases} \qquad \epsilon_j = \begin{cases} +1 & \text{if } 1 \leq j \leq 2J \\ -1 & \text{if } 2J+1 \leq j \leq 4J \end{cases} \tag{82}$$

□

Since we are assuming that for every fixed n, $\mathfrak{t}_{n,k}(q) = 0$ for large enough k, it follows that the support of $\mathfrak{t}_{n,k}(q)$ is of the form (n,k) where $k \in nP \cap \mathbb{Z}^r$ for a suitable rational convex polytope $P \subset \mathbb{R}^{r+1}$.

7.2. From general q-terms to Neumann-Zagier matrices.

In this section we will connect the Variational Equations (12) to Neumann-Zagier matrices. The latter were introduced in [NZ] as a combinatorial abstraction of the combinatorics of an ideal triangulation of a hyperbolic 3-manifold.

Given a matrix $r \times 2r$ A with integer entries, and a vector $\epsilon \in \{-1,1\}^r$, consider the system of r polynomial equations in the r variables $z = (z_1, \dots, z_r)$:

$$\prod_{j=1}^{r} z_i^{A_{i,j}} (1-z)^{A_{i,r+j}} = \epsilon_j, \qquad i = 1, \dots, r. \tag{83}$$

The above system of Equations are called *Gluing Equations* associated to the matrix A. In hyperbolic geometry, the Gluing Equations arise in deformation of the hyperbolic structure of a space given by the gluing of r ideal tetrahedra of shapes $z_1, \dots, z_r$. In that case, the rows of the matrix A span an isotropic subspace in $\mathbb{R}^{2r}$, which in addition is middle-dimensional, i.e., Lagrangian, see [NZ, Prop.2.3]. Such matrices are sometimes called *Neumann-Zagier matrices* in the literature. For the purposes of our paper, we will declare that A is a *Neumann-Zagier* matrix then its rows span an isotropic subspace in $\mathbb{R}^{2r}$.

Comparing the variational equations (12) associated to a general q-term $\mathfrak{t}$ as in (2) with the gluing equations (83) associated to a Neumann-Zagier matrix we observe that the only difference is in the presence of the monomials z^A for $A = A_j$, $j = 1, \dots, J$. There is a canonical way to accommodate for this difference, by simply introducing new variables z_{A_j} (for $j = 1, \dots, J$) and corresponding equations

$$z_{A_j} = z^{A_j}. \tag{84}$$

In other words, we have the following.

Lemma 7.2. *There exists a map:*

$$\mathrm{NZ} : \textit{General } q\textit{-terms} \longrightarrow \textit{Neumann-Zagier matrices}, \qquad \mathfrak{t} \mapsto \mathrm{NZ}_{\mathfrak{t}} \tag{85}$$

that fits into a commutative diagram:

$$\begin{tikzcd}
\textit{General } q\textit{-terms} \arrow[rr, "\mathrm{NZ}"] \arrow[dr, "\textit{Variational Equations}"'] & & \textit{Neumann-Zagier matrices} \arrow[dl, "\textit{Gluing Equations}"] \\
& \textit{Complex points of Varieties over } \mathbb{Q} &
\end{tikzcd}$$

Remark 7.3. More precisely, the commutative diagram above states that variety defined by the Variational Equations associated to $\mathfrak{t}$ is *birational* over $\mathbb{Q}$ to the variety defined by the Gluing equations of $\mathrm{NZ}_{\mathfrak{t}}$, via explicit maps.

Proof. Fix a general q-term $\mathfrak{t}_{n,k}$ as in (2), where $k = (k_1, \dots, k_r)$. Consider the variables z_i for $i = 0, \dots, r$ and add the variables z_{A_j} for $j = 1, \dots, J$. Consider the union of the Variational Equations (12) corresponding to $\mathfrak{t}$, together with the Equations (84) for the added variables.

With the notation of Equation (10), let us define the matrices M and N of size $(r+1) \times J$ matrices M as follows:

$$M_{i,j} = \epsilon_j v_i(A_j), \qquad M_{i,j} = v_i(A_j) \tag{86}$$

for $i = 0, \dots, r$, $j = 1, \dots, J$. Let us now define the matrix $\mathrm{NZ}_{\mathfrak{t}}$ of size $(r+1+J) \times (2r+2+2J)$ as follows:

$$\mathrm{NZ}_{\mathfrak{t}} := \begin{pmatrix} 2Q & 0_{(r+1)xJ} & 0_{(r+1)x(r+1)} & M \\ N^T & I_{J\times J} & 0_{Jx(r+1)} & 0_{JxJ} \end{pmatrix}, \tag{87}$$

where N^T denotes the transpose of N and the ordering of the columns of $\mathrm{NZ}_{\mathfrak{t}}$ is given by

$$\begin{pmatrix} z_i & z_{A_j} & 1 - z_i & 1 - z_{A_j} \end{pmatrix}.$$

It is easy to check that the Gluing Equations (83) of $NZ_{\mathfrak{t}}$ are identical to the Variational Equations (12) union Equations (84) are identical to the Variational of $\mathfrak{t}$. Moreover, it is easy to check that the rows of $\mathrm{NZ}_{\mathfrak{t}}$ is an isotropic subspace of $\mathbb{R}^{2r+2+2J}$. □

Remark 7.4. If M is a matrix of full rank, then $\mathrm{NZ}_{\mathfrak{t}}$ is Lagrangian.

Remark 7.5. Sometimes, we can associate Neumann-Zagier matrices of smaller size to a general q-term $\mathfrak{t}$. For example, when $A_j = A^{j'}$ then we can use only one added variable z_{A_j}.

8. For completeness

8.1. **Motivation for the special function Φ and the potential function.** Recall the special function Φ given by Equation (37). The next lemma is our motivation for introducing Φ. Kashaev informs us that this computation was well-known to Faddeev, and was a starting point in the theory of q-dilogarithm function; see [FK].

Lemma 8.1. *For every $\alpha \in (0, 1)$ we have:*

$$\prod_{k=1}^{[\alpha N]} (1 - e^{\frac{2\pi i k}{N}}) = e^{N\Phi(e^{2\pi i \alpha}) + O\left(\frac{\log N}{N}\right)}. \tag{88}$$

Proof. The proof is similar to the proof of [GL2, Prop.8.2], and follows from applying the *Euler-MacLauring summation formula*

$$\begin{aligned}\log\Big(\prod_{k=1}^{[\alpha N]}(1-e^{\frac{2\pi i k}{N}})\Big) &= \sum_{k=1}^{[\alpha N]}\log(1-e^{\frac{2\pi i k}{N}}) \\ &= \alpha N\int_0^1 \log(1-e^{2\pi i a x})dx + O\left(\frac{\log N}{N}\right),\end{aligned}$$

together with the fact that:

$$\int_0^1 \log(1-e^{2\pi i a x})dx = \frac{1}{2\pi i \alpha}\left(\frac{\pi^2}{6}-\mathrm{Li}_2(e^{2\pi i \alpha})\right).$$

□

Fix a special q-term $\mathfrak{t}_{n,k}(q)$ where $k=(k_1,\dots,k_r)$ and a positive natural number $N\in\mathbb{N}$. Fix also $w=(w_0,\dots,w_r)$ where $w_i\in(0,1)$ for $i=0,\dots,r$. Let us abbreviate $([w_0N],[w_1N],\dots,[w_rN])$ by $[wN]$. Lemma 8.1 implies the following.

Lemma 8.2. *With the above assumptions, we have:*

$$\log \mathfrak{t}_{[wN]} = e^{N V_{\mathfrak{t}}(e^{2\pi i w})+O\left(\frac{\log N}{N}\right)}. \tag{89}$$

This motivates our definition of the potential function.

8.2. **Laplace's method for a q-term.** There is an alternative way to derive the Variational Equations (12) from a general q-term $\mathfrak{t}$.

Since $\mathfrak{t}_{n,k}$ is q-hypergeometric, and $k=(k_1,\dots,k_r)$, it follows that for every $i=0,\dots,r$ we have

$$R_i(z_0,\dots,z_r,q) := \frac{\mathfrak{t}_{n,k_1,\dots,k_i+1,\dots,k_r}(q)}{\mathfrak{t}_{n,k_1,\dots,k_r}(q)} \in \mathbb{Q}(z_0,\dots,z_r,q)$$

where $z_i=q^{k_i}$ for $i=1,\dots,r$ and $z_0=q^n$. It is easy to see that the system of equations:

$$R_i(z_0,z_1,\dots,z_r)=1, \qquad i=0,\dots,r.$$

is identical to the system (12) of variational equations. In discrete math, the above system is known as *Laplace's method*.

8.3. **A comparison between the Bloch group and its extended version.** A comparison between the extended Bloch-Suslin complex and the Suslin complex is summarized in the following diagram with short exact rows and columns. The diagram is taken by combining [GZ, Sec.3] with [Ne, Thm.7.7], and using the map $\hat{\chi}$ from [GZ, Eqn.3.11].

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "e^{2\pi i \bullet}"] \arrow[d, "\hat{\psi}"] & \mathbb{C}^* \arrow[r] \arrow[d, "\hat{\chi}"] & \mathbb{C}/(\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\tau"] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\mathcal{B}(\mathbb{C})} \arrow[r] \arrow[d] & \widehat{\mathcal{P}(\mathbb{C})} \arrow[r, "\hat{\nu}"] \arrow[d] & \mathbb{C}\wedge\mathbb{C} \arrow[r] \arrow[d, "-\exp\wedge\exp"] & K_2(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{B}(\mathbb{C}) \arrow[r] \arrow[d] & \mathcal{P}(\mathbb{C}) \arrow[r, "\nu"] \arrow[d] & \mathbb{C}^*\wedge\mathbb{C}^* \arrow[r] \arrow[d] & K_2(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where

$$\begin{aligned} G \wedge G &= G \otimes_{\mathbb{Z}} G/(a \otimes b + b \otimes a) \\ \hat{\psi}(z) &= \hat{\chi}(e^{2\pi i z}) \\ \tau(z) &= z \wedge 2\pi i \\ (-\exp \wedge \exp)(a \wedge b) &= -(e^{2\pi i a} \wedge e^{2\pi i b}). \end{aligned}$$

In addition, we have the following useful Corollary, from [GZ, 3.14].

Corollary 8.3. *For $z \in \mathbb{C}$ we have:*

$$\frac{1}{(2\pi i)^2}\hat{R}(\hat{\chi}(e^{2\pi i z})) = z$$

The restriction

$$\hat{R} : \mathrm{Ker}(\widehat{\mathcal{B}(\mathbb{C})} \to \mathcal{B}(\mathbb{C})) \longrightarrow \mathbb{C}/\mathbb{Z}(2)$$

is 1-1.

9. Some questions

In this section we will list some problems that we hope to return in the near future.

The Bloch-Suslin complex (5) is just the tip of the mountain. Several authors have developed generalizations of the complex (5) in relation to special valued of L-functions and motivic cohomology. See for example, [BD, Go1, Go2, Za]. For $F = \mathbb{C}$, Suslin has computed in [Su3, Thm.4.9] that the torsion of $K_n(\mathbb{C})$ is given by:

$$\mathrm{Torsion}(K_n(\mathbb{C})) = \begin{cases} 0 & \text{if } n = \text{even} \\ \mathbb{Q}/\mathbb{Z} & \text{if } n = \text{odd} \end{cases}$$

Problem 9.1. Develop an extended version of the Bloch-Suslin complex for $\mathbb{C}$ whose kernel to the Bloch-Suslin complex matches the torsion of $K_n(\mathbb{C})$.

Without doubt, an obstacle to defining an extended version of the Bloch-Suslin complexes is to understand a proper generalization of the *Rogers* dilogarithm. For example, the Rogers dilogarithm $L(z)$ from (23) satisfies the following curious matrix relation:

$$e^{A(z)} = B(z)$$

where

$$A(z) = \begin{pmatrix} 0 & \mathrm{Log}(1-z) & \mathrm{Li}_2(z) \\ 0 & 0 & \mathrm{Log}(z) \\ 0 & 0 & 0 \end{pmatrix}, \qquad B(z) = \begin{pmatrix} 1 & \mathrm{Log}(1-z) & L(z) + \frac{\pi^2}{2} \\ 0 & 1 & \mathrm{Log}(z) \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix $A(z)$ generalizes to polylogarithms; see [BD].

Problem 9.2. Find a generalization of the Rogers dilogarithm and its functional equations.

Problem 9.3. Understand the Rogers dilogarithm and the 5-term relation from the Hodge theory point of view.

In all known examples of general q-terms $\mathfrak{t}$, the set of complex solutions $X_{\mathfrak{t}}$ of the Variational Equations (12) is finite.

Problem 9.4. Give sufficient conditions on a general q-term$\mathfrak{t}$ which imply that $X_{\mathfrak{t}}$ is finite and nonempty.

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