

# ASYMPTOTICS OF CLASSICAL SPIN NETWORKS

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ABSTRACT. A spin network consists of a cubic ribbon graph (i.e., an abstract trivalent graph with a cyclic ordering of the vertices around each edge) and an admissible coloring of its edges by natural numbers. The evaluation of a spin network, with our normalization, is an integer number. The main results of our paper are: (a) an existence theorem for the asymptotics of evaluations of arbitrary spin networks (using the theory of  $G$ -functions), (b) a rationality property of the generating series of all evaluations with a fixed underlying graph (using the combinatorics of the chromatic evaluation of a spin network), (c) a complete study of the asymptotics of the so-called  $6j$ -symbols in all cases: Euclidean, Plane or Minkowskian (using the theory of Borel transform), (d) an assignment of two numerical invariants (a spectral radius and a number field) to spin networks and cubic graphs and (e) an explicit illustration of our results for some Euclidean, Plane and Minkowskian  $6j$ -symbols (using the constructive Wilf-Zeilberger method).

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## 1. INTRODUCTION

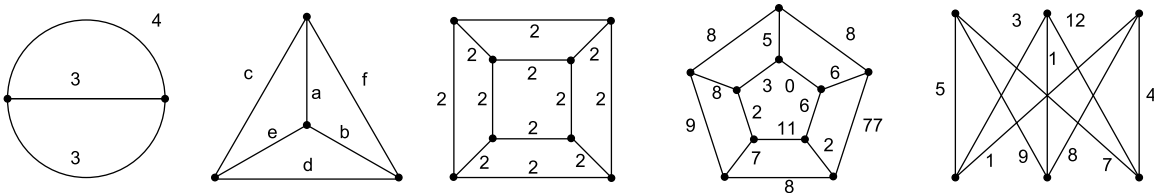
1.1. **Spin networks and their evaluations.** The paper is concerned with

- (a) an existence theorem for the asymptotics of evaluations of arbitrary spin networks (using the theory of  $G$ -functions),
- (b) a rationality property of the generating series of all evaluations with a fixed underlying graph (using the combinatorics of the chromatic evaluation of a spin network),
- (c) a complete study of the asymptotics of the so-called  $6j$ -symbols in all cases: Euclidean, Plane or Minkowskian (using the theory of Borel transform),
- (d) the assignment of two numerical invariants to spin networks and cubic graphs, the spectral radius and a number field (see Definition 2.3),
- (e) an explicit illustration of our results for the regular and some flat  $6j$ -symbols (using the constructive WZ method).

A (classical) *spin network*  $(\Gamma, \gamma)$  consists of a *cubic ribbon graph*  $\Gamma$  (i.e., an abstract trivalent graph with a cyclic ordering of the edges at each vertex) and a coloring  $\gamma$  of its set of edges by natural numbers.

According to Penrose, spin networks correspond to a diagrammatic description of tensors of representations of  $SU(2)$ , and their evaluation is a contraction of the above tensors. Spin networks originated in Racah's work in atomic spectroscopy in the late forties [Ra] and in the work of Wigner in the fifties [Wi] and their evaluations (exact or asymptotic) is a useful and interesting topic studied by Ponzano-Regge, Biedenharn-Louck and many others; see [BL1, BL2, PR, VMK]. In the past three decades, spin networks have been used in relation to classical and quantum gravity and angular momentum in 3-dimensions; see [EPR, Pe1, Pe2, RS]. On the mathematical side,  $q$ -deformations of spin networks (so called quantum spin networks) appeared in the eighties in the work of Kirillov-Reshetikhin [KR]. *Quantum spin networks* are *knotted framed trivalent graphs* embedded in 3-space with a cyclic ordering of the edges near every vertex, and their evaluations are rational functions of a variable  $q$ . The quantum  $3j$  and  $6j$ -symbols are the building blocks for topological invariants of closed 3-manifolds in the work of Turaev-Viro [TV, Tu]. Quantum spin networks are closely related to a famous invariant of knotted 3-dimensional objects, the celebrated *Jones polynomial*, [J]. A thorough discussion of quantum spin networks and their relation to the Jones polynomial and the *Kauffman bracket* is given in [KL] and [CFS].

Aside from the above applications, spin networks and their evaluations have an intrinsic interest, they are formulated in terms of elementary combinatorial data and offer many challenges even for relatively simple networks such as the cube. Some examples of spin networks that will be discussed in the paper are shown in Figure 1.



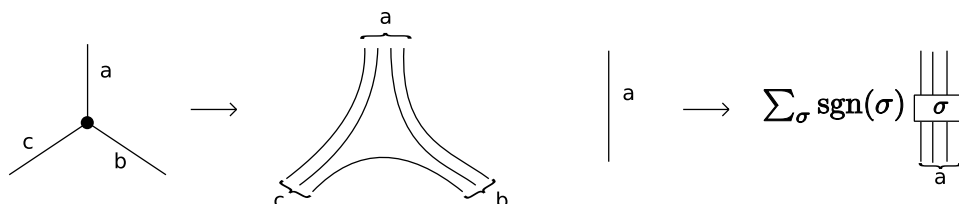
**Figure 1.** From left to right: The theta or  $3j$ -symbol, the tetrahedron or  $6j$ -symbol, the Cube, the 5-sided drum and the complete bipartite graph  $K_{3,3}$  or  $9j$ -symbol. The cyclic order of the edges around each vertex is counterclockwise.

Spin networks with underlying graph  $\Theta$  or a tetrahedron are usually called  $3j$ -symbols and  $6j$ -symbols respectively and they form the building blocks for the evaluation of any spin network; see for example Section 6.1 or [BL1, BL2, PR]. We now discuss the evaluation of a spin network, introduced by Penrose in [Pe1, Pe2].

**Definition 1.1.** (a) We say a spin network is *admissible* when the sum of the three labels  $a, b, c$  around every vertex is even and  $a, b, c$  satisfy the triangle inequalities:  $|a - b| \leq c \leq a + b$ .

(b) The evaluation  $\langle \Gamma, \gamma \rangle^P$  of a spin network  $(\Gamma, \gamma)$  is defined to be zero if it is not admissible. An admissible spin network is evaluated by the following algorithm.

- Use the cyclic ordering to thicken the vertices into disks and the edges into untwisted bands.
- Replace the vertices and edges by the linear combinations of arcs as shown in Figure 2.
- Finally the resulting linear combination of closed loops is evaluated by assigning the value  $(-2)^n$  to a term containing  $n$  loops.



**Figure 2.** The rules for evaluating a spin network. Replace vertices and edges by linear combinations of arcs and count the number of closed loops.

For example, in Figure 1 the  $\Theta$  network and the Cube are admissible, the 5-sided drum and the  $9j$ -symbol are not. Note that the admissibility condition is equivalent to saying that the strands can be connected at each vertex as in Figure 2. Note also that cubic ribbon graphs  $\Gamma$  are allowed to have multiple edges, loops and may be disconnected with components that contain no vertices. In addition,  $\Gamma$  may be and non-planar (contrary to the requirement of many authors [We, Mou, KL]), as long as one fixes a cyclic ordering of the edges at each vertex. The latter condition is implicit in [Pe1]. It turns out that a different cyclic ordering on a vertex of a spin network effects its evaluation by a sign; see Lemma 6.1 below.

**1.2. A fundamental problem.** It is easy to see that if  $(\Gamma, \gamma)$  is an admissible spin network and  $n$  is a natural number, then  $(\Gamma, n\gamma)$  is admissible as well. A fundamental problem is to study the asymptotic behavior of the sequence of evaluations  $\langle \Gamma, n\gamma \rangle^P$  when  $n$  is large. This problem actually consists of three parts.

**Problem 1.** Fix an admissible spin network  $(\Gamma, \gamma)$ .

- Prove the existence of an asymptotic expansion of the sequence  $\langle \Gamma, n\gamma \rangle^P$  when  $n$  is large.
- Compute the asymptotic expansion of  $\langle \Gamma, n\gamma \rangle^P$ .
- Identify the terms in the asymptotic expansion with geometric invariants of the spin network.

Although any spin network can be evaluated in terms of  $3j$  and  $6j$ -symbols, a positive solution to parts (a), (b) or (c) of Problem 1 for  $3j$  and  $6j$ -symbols does *not* imply a solution to for arbitrary spin networks.

Problem 1 is motivated by the belief that the quantum mechanics of particles with large spin will approximate the classical theory. There is a large physics literature for Part (b); see for example [VMK]. On the other hand, the literature for Part (a) is relatively new and short, and to the best of our knowledge concerns only  $3j$  and  $6j$ -symbols with certain labellings. Roberts used geometric quantization techniques to prove the leading asymptotic behavior of  $6j$ -symbols in the so-called Euclidean case; see [Rb1, Rb2].

In order to get an exponentially (and not factorially) growing asymptotic expansion we will rescale the evaluations of spin networks by factorials of linear forms on the labels of their edges as follows.

**Definition 1.2.** We define the *standard normalization* of a spin network evaluation to be

$$(1) \quad \langle \Gamma, \gamma \rangle = \frac{1}{\mathcal{I}!} \langle \Gamma, \gamma \rangle^P$$

where  $\mathcal{I}!$  is defined to be the product

$$(2) \quad \mathcal{I}! = \prod_{v \in V(\Gamma)} \left( \frac{-a_v + b_v + c_v}{2} \right)! \left( \frac{a_v - b_v + c_v}{2} \right)! \left( \frac{a_v + b_v - c_v}{2} \right)!$$

where  $a_v, b_v, c_v$  are the colors of the edges emanating from vertex  $v$ , and  $V(\Gamma)$  is the set of vertices of  $\Gamma$ .

There are many useful properties of this evaluation:

- $\langle G, \gamma \rangle$  is still an integer (see Theorem 5),
- the sequence  $\langle \Gamma, n\gamma \rangle$  is exponentially bounded (see Theorem 1),
- the spin generating series is a rational function (see Theorem 5 below).

## 2. EXISTENCE OF ASYMPTOTICS

**2.1. Asymptotics via  $G$ -functions.** Our first result is a complete solution of Part (a) of Problem 1 for all spin networks, using the theory of  $G$ -functions. To state it, we need to recall the notion of a sequence of Nilsson type [Ni].

**Definition 2.1.** We say that a sequence  $(a_n)$  is of *Nilsson type* if it has an *asymptotic expansion* of the form

$$(3) \quad a_n \sim \sum_{\lambda, \alpha, \beta} \lambda^n n^\alpha (\log n)^\beta S_{\lambda, \alpha, \beta} h_{\lambda, \alpha, \beta}(1/n)$$

where

- the summation is over a finite set,
- the *growth rates*  $\lambda$  are algebraic numbers of equal magnitude,
- the *exponents*  $\alpha$  are rational and the *nilpotency exponents*  $\beta$  are natural numbers,
- the *Stokes constants*  $S_{\lambda, \alpha, \beta}$  are complex numbers,
- the formal power series  $h_{\lambda, \alpha, \beta}(x) \in K[[x]]$  are Gevrey-1 (i.e., the coefficient of  $x^n$  is bounded by  $C^n n!$  for some  $C > 0$ ),
- $K$  is a *number field* generated by the coefficients of  $h_{\lambda, \alpha, \beta}(x)$  for all  $\lambda, \alpha, \beta$ .

The above definition uses the notion of an asymptotic expansion in the sense of Poincaré, given in [O, Sec.1]. In the special case where there is only one growth rate, the notation

$$a_n \sim \lambda^n n^\alpha (\log n)^\beta \sum_{k=0}^{\infty} \frac{\mu_k}{n^k}$$

means that for every  $N \in \mathbb{N}$  we have

$$\lim_{n \rightarrow \infty} n^N \left( a_n \lambda^{-n} n^{-\alpha} (\log n)^{-\beta} - \sum_{k=0}^{N-1} \frac{\mu_k}{n^k} \right) = \mu_N.$$

In Section 7 we will prove the following.

**Theorem 1.** *For every spin network  $\langle \Gamma, \gamma \rangle$ , the sequence  $\langle \Gamma, n\gamma \rangle$  is of Nilsson type.*

Theorem 1 follows from the theory of  $G$ -functions by combining Theorems 2 and 3 below.  $G$ -functions were introduced by Siegel ([Si]) in relation to transcendence problems in number theory. Many of their arithmetic and algebraic properties were established by André in [An].  $G$ -functions appear naturally in Geometry (as Variations of Mixed Hodge Structures), in Arithmetic and most recently in Enumerative Combinatorics. For a detailed discussion, see [Ga4] and references therein. Let us recall the notion of a  $G$ -function.

**Definition 2.2.** We say that series  $G(z) = \sum_{n=0}^{\infty} a_n z^n$  is a  *$G$ -function* if

- the coefficients  $a_n$  are algebraic numbers,
- there exists a constant  $C > 0$  so that for every  $n \in \mathbb{N}$  the absolute value of every conjugate of  $a_n$  is less than or equal to  $C^n$ ,
- the common denominator of  $a_0, \dots, a_n$  is less than or equal to  $C^n$ ,
- $G(z)$  is holonomic, i.e., it satisfies a linear differential equation with coefficients polynomials in  $z$ .

The following theorem is proven in Section 7. The proof is elementary given the results of [Ga4] and the fact that an evaluation  $\langle \Gamma, n\gamma \rangle$  can be written as a multi-dimensional combinatorial sum of a balanced hypergeometric term.

**Theorem 2.** *The generating function*

$$(4) \quad F_{\Gamma, \gamma}(z) = \sum_{n=0}^{\infty} \langle \Gamma, n\gamma \rangle z^n$$

*of an arbitrary spin network  $(\Gamma, \gamma)$  is a  $G$ -function.*

The next result follows by a combination of results of André, Katz and Chudnovsky-Chudnovsky; see [An] and also [Ga4, Sec.2].

**Theorem 3.** *If  $G(z) = \sum_{n=0}^{\infty} a_n z^n$  is a  $G$ -function, then  $(a_n)$  is a sequence of Nilsson type.*

Every  $G$ -function has two basic invariants: an analytic and an arithmetic one:

- the radius of convergence at  $z = 0$ , which equals to  $|\lambda|^{-1}$  where  $\lambda$  is a growth rate of  $(a_n)$  as in Definition 2.1,
- a number field  $K$  corresponding to  $(a_n)$  as in Definition 2.1.

Theorem 2 motivates the following invariants of a spin network: its spectral radius and its number field.

**Definition 2.3.** Let  $(\Gamma, \gamma)$  be a spin network.

- (a) The *spectral radius*  $\rho_{\Gamma, \gamma}$  is defined to be the inverse of the radius of convergence of the generating function  $F_{\Gamma, \gamma}(z)$  at  $z = 0$ .
- (b) The *number field*  $K_{\Gamma, \gamma}$  is defined to be the number field associated to the  $G$ -function (4).
- (c) The *spectral radius*  $\rho_{\Gamma}$  and the number field  $K_{\Gamma}$  of a cubic ribbon graph is defined to be  $\rho_{\Gamma, 2}$  and  $K_{\Gamma, 2}$  respectively, where 2 denotes the admissible coloring of all edges of  $\Gamma$  by 2.

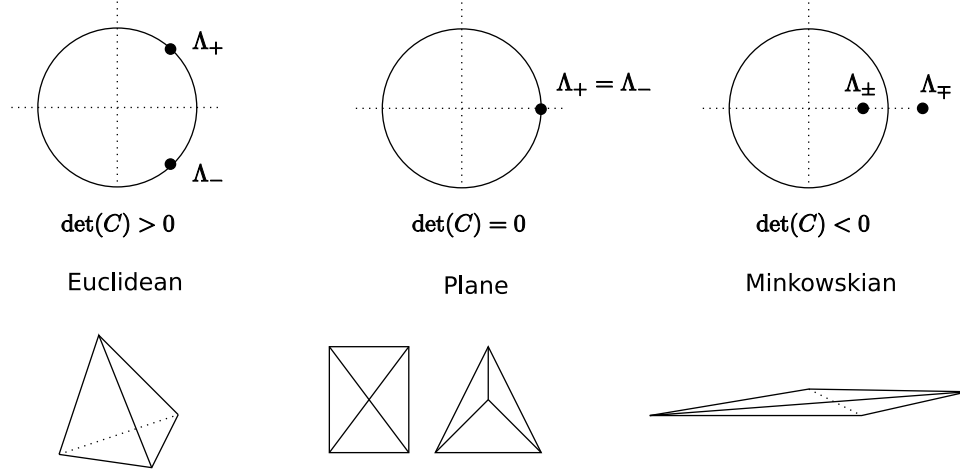
**2.2. Computation of asymptotics.** The proof of Theorem 1 is not constructive. We know of several different methods for computing the asymptotic expansions of spin network evaluations. We will list them here briefly and illustrate them in the following sections.

- (a) The method of *Borel transform* for one-dimensional sums of balanced hypergeometric terms. This method has been developed for one-dimensional sums in [CG1, CG2, Ga3]) and gives exact formulas for the complete asymptotic expansion of such sums. The method ought to extend to multi-dimensional sums, but this still needs to be developed. Since  $6j$ -symbols are given by one-dimensional sums of a balanced hypergeometric term (in all cases, Euclidean, Plane or Minkowskian), this allows us to compute exactly the leading asymptotic behavior of  $6j$ -symbols.
- (b) The method of  *$G$ -functions*. We mentioned already that the generating series  $F_{\Gamma, \gamma}(z)$  of (4) is a  $G$ -function in a non-constructive way, but more is actually true. In Theorem 6 we show that  $F_{\Gamma, \gamma}(z)$  is the diagonal of an explicit rational function. Thus, the asymptotics of the Taylor coefficients  $\langle \Gamma, n\gamma \rangle$  of  $F_{\Gamma, \gamma}(z)$  can be computed in principle via Hodge theory, along the lines of [BK]; see also [PW1, PW2]. Unfortunately, Pemantle-Wilson's results require genericity assumptions that do not apply even in the case of the  $K_{3,3}$  or the Cube spin network.
- (c) The method of Wilf-Zeilberger. The fundamental theorem of Wilf-Zeilberger implies that for an arbitrary spin network  $(\Gamma, \gamma)$ , the sequence  $\langle \Gamma, n\gamma \rangle$  is a multi-dimensional sum of a hypergeometric term, and thus, holonomic, i.e., it satisfies a recursion relation with coefficients polynomials in  $n$ , [WZ1]. The WZ method is constructive, and has been implemented in several platforms, [PWZ, PaRi, WZ1]. WZ has been implemented for multidimensional sums, but in practice it works best for one or two dimensional sums. Using the WZ method, we compute the asymptotics of three simple examples, a Euclidean  $6j$ -symbol, a Plane one and a Minkowskian  $6j$ -symbol, in Section 10.
- (d) The ansatz for the asymptotics of hypergeometric multi-dimensional sums. In [Ga2] the first author developed a constructive ansatz for locating the singularities of the generating series of balanced hypergeometric multi-dimensional sums. The conjecture has been confirmed for one-dimensional sums using the theory of Borel transform. For higher dimensional sums the conjecture requires a genericity assumption (finiteness of the solution set of the so-called Variational Equations) which does not seem to hold for spin networks with more than six edges. A proof of the ansatz in the higher dimensional case is expected to follow by developing further the theory of Borel transform.

The practical limitation of these methods is discussed in Section 12.

3. ASYMPTOTICS OF  $6j$ -SYMBOLS

**3.1. The geometry of the tetrahedron and asymptotics of the  $6j$ -symbol.** In the special case of the tetrahedron graph we present a complete solution to Parts (b) and (c) of Problem 1 using the method of Borel transform. We can explicitly compute the asymptotic expansions for all possible colorings, even in the degenerate and non-physical cases. Moreover the quantities in the asymptotic expansion can be expressed as geometric properties of the dual tetrahedron graph interpreted as a metric polyhedron.



**Figure 3.** The three realizations of the dual metric tetrahedron depending on the sign of the Cayley-Menger determinant  $\det(C)$ . Also depicted are the two exponential growth factors  $\Lambda_{\pm}$  of the  $6j$ -symbol.

In this section  $(\triangle, \gamma)$  denotes the tetrahedral spin network colored by an admissible coloring  $\gamma$ . Consider the planar dual graph  $\nabla$  (also a tetrahedron) with the same labelling on the edges. Regarding the labels of  $\nabla$  as edge lengths one may ask whether  $(\nabla, \gamma)$  can be realized as a metric Euclidean tetrahedron with these edge lengths. The admissibility of  $(\triangle, \gamma)$  implies that all faces of the dual tetrahedron  $\nabla$  satisfy the triangle inequality. A well-known theorem of metric geometry [Bl] implies that  $(\nabla, \gamma)$  can be realized in *exactly one* of three flat geometries

- (a) Euclidean 3-dimensional space  $\mathbb{R}^3$
- (b) Minkowskian space  $\mathbb{R}^{2,1}$
- (c) Plane Euclidean  $\mathbb{R}^2$ .

Moreover, the above trichotomy is decided by the sign of the *Cayley-Menger determinant*  $\det(C)$  which is a degree six polynomial in the six edge labels; see Definition 9.1. Without loss of generality, we will assume that  $(\triangle, \gamma)$  is *non-degenerate* in the sense that all faces of  $\nabla$  are two dimensional (i.e., the triangle inequalities are strict). In the degenerate case, the evaluation of  $6j$ -symbol is a ratio of factorials, whose asymptotics are easily obtained from Stirling's formula. Notice that  $(\triangle, \gamma)$  can be degenerate only in the Plane and Minkowskian cases.

**3.2. Numerical invariants of tetrahedra.** Every sequence of Nilsson type comes with its own numerical invariants: its growth rates, exponents, nilpotency exponents, Stokes constants and number field; see Definition 2.1. Theorem 1 implies that  $6j$ -symbols have such invariants. In this section we will define a set of numerical invariants of tetrahedra and in the next section we will identify those with the numerical invariants of the corresponding  $6j$ -symbols. In this section, we will fix a nondegenerate tetrahedral spin network  $(\triangle, \gamma)$ .

**Definition 3.1.** (a) Let  $\theta_a$  be the *exterior dihedral angle* corresponding to edge  $a$  of the metric realization of  $(\nabla, \gamma)$ .

(b) Consider the growth rates  $\Lambda_{\pm}$  defined by:

$$(5) \quad \Lambda_{\pm} = e^{\pm i \sum_a \theta_a \frac{\alpha_a}{2}}.$$

(c) Consider the number field

$$(6) \quad K = \mathbb{Q}(\sqrt{-\det(C)})$$

The next Lemma, proven in Section 9.2, explains the top part of Figure 3.

**Lemma 3.2.** *The growth rates  $\Lambda_{\pm}$  have the following properties.*

(a)  $\Lambda_+ \Lambda_- = 1$

(b) *In the Euclidean case we have additionally that  $\Lambda_- = \overline{\Lambda_+}$  and  $\Lambda_+ \neq \Lambda_-$  and consequently  $|\Lambda_+| = |\Lambda_-| = 1$ , and  $\Lambda_{\pm} \neq \pm 1$ .*

(c) *In the Plane case, we have additionally that  $\Lambda_+ = \Lambda_-$  and consequently  $\Lambda_{\pm} \in \{-1, 1\}$ .*

(d) *In the Minkowsian case, we have additionally that  $\Lambda_{\pm} \in \mathbb{R} \setminus \{0\}$ . Exactly one of  $\Lambda_{\pm}$  is inside and one is outside the unit circle. Let  $\Lambda$  denote the one inside the unit circle.*

**Definition 3.3.** We define the exponents  $\alpha_{\pm}$  by

$$(7) \quad (\alpha_+, \alpha_-) = \begin{cases} (-\frac{3}{2}, -\frac{3}{2}) & \text{if } \gamma \text{ is Euclidean or Minkowsian} \\ (-\frac{4}{3}, -\frac{5}{3}) & \text{if } \gamma \text{ is Plane.} \end{cases}$$

**Definition 3.4.** (a) If  $\gamma$  is Euclidean, we define

$$(8) \quad S_{\pm} = \frac{e^{\pm i(\sum_a \frac{\alpha_a}{2} + \frac{\pi}{4})}}{\sqrt{6\pi \text{Vol}}}$$

(b) If  $\gamma$  is Minkowsian, we define

$$(9) \quad S_{\pm} = \mp \frac{e^{\pm i(\sum_a \frac{\alpha_a}{2})}}{\sqrt{6\pi \text{Vol}}}$$

(c) If  $\gamma$  is Plane, we define

$$(10) \quad S_{\pm} = \frac{\Gamma(4/3)(12A)^{1/3}}{\pi \prod |B_j|^{1/6}}$$

The definition of  $A$  and  $B_j$  (homogeneous polynomials in  $\gamma$  of degree 2 and 3 respectively) in the Plane case can be found in Section 9.2.

**3.3. Asymptotics of  $6j$ -symbols.** In this section we will express the asymptotics of the  $6j$  symbols (using the minor change  $\langle \cdot \rangle^U$  in normalization explained in Definition 9.4) in terms of the numerical invariants of the previous section.

**Theorem 4.** *Fix a nondegenerate tetrahedron  $(\triangle, \gamma)$ .*

(a) *If  $\gamma$  is Euclidean or Plane, there exist  $h_{\pm} \in K[[x]]$  so that*

$$(11) \quad \langle \triangle, n\gamma \rangle^U \sim \Lambda_+^n n^{\alpha_+} S_+ h_+ \left(\frac{1}{n}\right) + \Lambda_-^n n^{\alpha_-} S_- h_- \left(\frac{1}{n}\right)$$

(b) *If  $\gamma$  is Minkowsian, let  $S$  be the Stokes constant corresponding to  $\Lambda$ . There exists  $h \in K[[x]]$  so that*

$$\langle \triangle, n\gamma \rangle^U \sim \Lambda^n n^{\alpha} S h \left(\frac{1}{n}\right)$$

Moreover,  $K$  is the associated number field.

Equation (11) implies in the Euclidean case that

$$\langle \triangle, n\gamma \rangle^U = \frac{\sqrt{2}}{\sqrt{3\pi n^3 \text{Vol}}} \cos \left( n \sum_a a \frac{\theta_a}{2} + \sum_a \frac{\theta_a}{2} + \frac{\pi}{4} \right) \left( 1 + O\left(\frac{1}{n}\right) \right)$$

which is the well-known *Ponzano-Regge formula* for the asymptotics of Euclidean  $6j$ -symbols. In the Euclidean and Minkowskian cases these asymptotic formulas were conjectured by Ponzano-Regge [PR]. An elementary but non-rigorous proof of the Euclidean case was given in [Gu]. The Euclidean case was rigorously proven by [Rb1] but there was no proof for the Minkowskian case. The formula for the Plane case is new.

*Remark 3.5.* Notice that exponents  $\alpha_{\pm}$  in the Plane case of Theorem 4 is not a half-integer but rather *the third of an integer*. This phenomenon (accompanied by the collision of the two points in the Plane case of Figure 3) is natural from the point of view of local monodromy of  $G$ -functions, and is often overlooked in the physics literature.

An illustration of Theorem 4 in all three geometries is given in Section 10, using the WZ method.

*Remark 3.6.* Using Theorem 4, we can compute explicitly the leading asymptotics of an infinite family of spin networks whose underlying graphs can be created from the tetrahedron graph by repeatedly replacing a vertex by a triangle. As follows from the recoupling theory discussed in Section 6 the corresponding spin network evaluation will change by multiplication by a certain  $6j$ -symbol and a theta. We thus find the asymptotics for all these networks. In particular, this leads to spin networks with an arbitrary number of growth rates, and with associated number fields which are the composite of an arbitrary number of quadratic number fields.

#### 4. A GENERATING FUNCTION FOR SPIN NETWORK EVALUATIONS

In this section we discuss the generating series  $S_{\Gamma}$  of a spin network evaluation for a fixed underlying graph  $\Gamma$ . As it turns out,  $S_{\Gamma}$  is a rational function. Among other things, this implies that the  $G$ -functions of Theorem 2 come from geometry and more precisely from a Variation of Mixed Hodge structures, much in the spirit of Bloch-Kreimer [BK].

The next definition introduces the generating series  $S_{\Gamma}$  of a fixed graph  $\Gamma$  with cyclic ordering.  $S_{\Gamma}$  is a formal power series in commuting variables  $z_e$  where  $e \in E(\Gamma)$ , the set of edges of  $\Gamma$ .

**Definition 4.1.** The *spin generating function*  $S_{\Gamma}$  of a cubic ribbon graph  $\Gamma$  is defined to be

$$(12) \quad S_{\Gamma} = \sum_{\gamma} \langle \Gamma, \gamma \rangle \prod_{e \in E(\Gamma)} z_e^{\gamma_e} \in \mathbb{Q}[[z_e, e \in E(\Gamma)]]$$

where the sum ranges over all colorings  $\gamma$  of the graph  $\Gamma$ .

We will express the spin generating function of an arbitrary graph in terms of the curve polynomial that we will define now. A *curve* in a graph  $\Gamma$  is a 2-regular subgraph. By definition the empty set is also a curve. The set of all curves in  $\Gamma$  is called  $C_{\Gamma}$ .

**Definition 4.2.** Given a cubic ribbon graph  $\Gamma$  and  $X \subset C_{\Gamma}$  we define

(a) a polynomial

$$(13) \quad P_{\Gamma, X} = \sum_{c \in C_{\Gamma}} \epsilon_X(c) \prod_{e \in c} z_e.$$

where  $\epsilon_X(c) = -1$  (resp. 1) when  $c \in X$  (resp.  $c \notin X$ ).

(b) We will call  $P_{\Gamma, \emptyset}$  the *curve polynomial* of  $\Gamma$ .

With the above notation, we define  $\iota(X)$  to be the minimal number of intersections when  $X$  is drawn on the thickened ribbon graph  $\Gamma$  and

$$(14) \quad a_X = \frac{1}{2^{|C_{\Gamma}|}} \sum_{Y \subset C_{\Gamma}} (-1)^{|X \cap Y| + \iota(Y)}.$$

The next theorem proves the rationality of  $S_\Gamma$ .

**Theorem 5.** *For every cubic ribbon graph  $\Gamma$  we have*

$$(15) \quad S_\Gamma = \sum_{X \subset C_\Gamma} a_X P_{\Gamma, X}^{-2} \in \mathbb{Z}[[z_\epsilon, \epsilon \in E(\Gamma)]] \cap \mathbb{Q}(z_\epsilon, \epsilon \in E(\Gamma)).$$

**Corollary 4.3.** *For every spin network  $(\Gamma, \gamma)$ , the evaluation  $\langle \Gamma, \gamma \rangle$  is an integer number.*

**Corollary 4.4.** [We] *When  $\Gamma$  is planar with the counterclockwise orientation, then  $a_X = 0$  unless  $X$  is empty, in which case  $a_\emptyset = 1$ . It follows that*

$$S_\Gamma = \frac{1}{P_{\Gamma, \emptyset}^2}$$

recovering an earlier theorem by Westbury [We].

The next theorem shows that the generating series  $F_{\Gamma, \gamma}(z)$  of (4) is a *diagonal* of a rational function. Let us recall the notion of a diagonal.

**Definition 4.5.** Given a power series  $f(x_1, \dots, x_r) \in \mathbb{Q}[[x_1, \dots, x_r]]$  and an exponent  $I = (i_1, \dots, i_r) \in \mathbb{N}_+^r$ , we define the  $I$ -diagonal of  $f$  by

$$(16) \quad (\Delta_I f)(z) = \sum_{n=0}^{\infty} [x^{nI}](f) z^n \in \mathbb{Q}[[z]]$$

where  $[x^{nI}](f)$  denotes the coefficient of  $x_1^{ni_1} \dots x_r^{ni_r}$  in  $f$ .

**Theorem 6.** *For every spin network  $(\Gamma, \gamma)$  we have*

$$(17) \quad F_{\Gamma, \gamma} = \Delta_\gamma S_F$$

Consequently, the  $G$ -function  $F_{\Gamma, \gamma}(z)$  is the diagonal of a rational function, and thus it comes from geometry; see Lemma 7.4 below.

We end this section with the following curious proposition.

**Proposition 4.6.** *If  $\Gamma$  is a 2-connected cubic ribbon graph, then the evaluations  $\langle \Gamma, \gamma \rangle$  for all  $\gamma$  uniquely determine  $\Gamma$  as an abstract trivalent graph. Moreover, we only need to use the colors 0 and 1.*

## 5. PLAN OF THE PAPER

In Section 6 we discuss the recoupling method for evaluating spin networks which reduces the evaluation of an arbitrary spin network to the  $3j$  and  $6j$ -symbols. Using this method, and an explicit formula for the  $3j$  and  $6j$ -symbols, we evaluate the Drum and the  $K_{3,3}$  graphs.

In Section 7 we prove that the evaluation of a spin network is given by a multi-dimensional sum of a balanced hypergeometric term. This, together with the results of [Ga4], implies Theorem 2 and Theorem 1.

In Section 8 we apply techniques of Borel transform (developed in [CG1, CG2, Ga3]) that give exact formulas for the leading asymptotic behavior of one-dimensional sums of hypergeometric terms. Since  $6j$ -symbols are given by such sums, this allows us to compute exactly the leading asymptotic behavior of  $6j$ -symbols.

Section 9 identifies the combinatorial leading asymptotic behavior with geometric quantities associated to the metric tetrahedron in all cases, Euclidean, Plane or Minkowskian.

In Section 10 we present an alternative constructive method to compute the asymptotic behavior of evaluations of spin networks, namely the WZ method.

In Section 11 we discuss in detail the chromatic evaluation of spin networks and obtain the integrality and rationality results of the spin generating series.

**5.1. Acknowledgment.** The results were conceived in a workshop in Aarhus, Denmark, and presented in HaNoi, Vietnam and Strasbourg, France in the summer of 2007. The authors wish to thank the organizers for their hospitality. S.G. wishes to thank D. Zeilberger for many enlightening conversations.

## 6. EVALUATION OF SPIN NETWORKS

We start by recording some elementary facts about spin network evaluations.

**Lemma 6.1.** *Let  $(\Gamma, \gamma)$  be a spin network.*

- (a) *Inserting a vertex colored  $(0, a, a)$  in the interior of an edge of  $\Gamma$  colored  $a$  does not change the evaluation of the spin network.*
- (b) *Changing the cyclic ordering at a vertex whose edges are colored  $a, b, c$  changes the evaluation by a sign  $(-1)^{(a(a-1)+b(b-1)+c(c-1))/2}$*

*Proof.* (a) The chosen normalization introduces an extra factor  $1/a!$  for the new vertex labeled  $(0, a, a)$ , while it follows from the definition that one also inserts an extraneous summation over permutations in the pre-existing edge labeled  $a$ . Since  $\sum_{\sigma \in S_a} \sum_{\tau \in S_a} \text{sign}(\sigma)\sigma \text{sign}(\tau)\tau = a! \sum_{\sigma \in S_a} \text{sign}(\sigma)\sigma$  the evaluation is unchanged.

(b) Changing the cyclic order at a vertex with edge labels  $a, b, c$  has the following effect. The alternating sum at each of the adjacent edges is multiplied by the permutation that turns the arcs in the edge by 180 degrees. This element has sign  $a(a-1)/2$  in  $S_a$ .  $\square$

As a consequence of part (a) of the above lemma, an edge labeled 0 in a spin network can be removed without affecting the evaluation.

There is an alternative *bracket normalization*  $\langle \Gamma, \gamma \rangle^B$  of the evaluation of a spin network  $(\Gamma, \gamma)$  which agrees with a specialization of the *Jones polynomial* and the *Kauffman bracket*.

**Definition 6.2.** The *bracket normalization* of a spin network  $(\Gamma, \gamma)$  is defined by

$$(18) \quad \langle \Gamma, \gamma \rangle^B = \frac{1}{\mathcal{E}!} \langle \Gamma, \gamma \rangle^P$$

where

$$(19) \quad \mathcal{E}! = \prod_{e \in E(\Gamma)} \gamma(e)!$$

This normalization has the property that it coincides with the Kauffman bracket (Jones polynomial) of a quantum spin network evaluated at  $A = -1$  [KL]. However,  $\langle \Gamma, \gamma \rangle^B$  is not necessarily an integer number, and the analogous generating series does not satisfy the rationality property of Theorem 5.

*Proof.* (of Proposition 4.6) For convenience we work with the unnormalized evaluation  $\langle \Gamma, \gamma \rangle^P$ . A 2-connected graph is determined up to isomorphism by its 2-regular subgraphs, simply take a polygon for every 2-regular subgraph and identify equal edges [Di]. To decide whether a set  $X$  of edges forms a 2-regular subgraph, let  $\gamma$  be the coloring that assigns 1 to members of  $X$  and 0 to the other edges. By definition of the spin network evaluation  $X$  is an  $n$  component 2-regular subgraph if and only if  $\langle \Gamma, \gamma \rangle^P = (-2)^n$ .  $\square$

**6.1. Evaluation of spin networks by recoupling.** In this subsection we describe a way of evaluating spin networks by recoupling. We will reduce the evaluation of spin networks to multi-dimensional sums of  $6j$  and  $3j$ -symbols. The value of the  $6j$  and  $3j$ -symbols is given by the following lemma of [KL] and [We], using our normalization.

**Lemma 6.3.** (a) Let  $(\triangle, \gamma)$  denote a tetrahedron labeled and oriented as in Figure 1 or Figure 5 (left) with  $\gamma = (a, b, c, d, e, f)$ . Its normalized evaluation is given by

$$(20) \quad \langle \triangle, \gamma \rangle = \sum_{k=\max T_i}^{\min S_j} (-1)^k (k+1) \binom{k}{S_1 - k, S_2 - k, S_3 - k, k - T_1, k - T_2, k - T_3, k - T_4}$$

where, as usual

$$\binom{a}{a_1, a_2, \dots, a_r} = \frac{a!}{a_1! \dots a_r!}$$

denotes the multinomial coefficient when  $a_1 + \dots + a_r = a$ , and  $S_i$  are the half sums of the labels in the three quadrangular curves in the tetrahedron and  $T_j$  are the half sums of the three edges emanating from a given vertex. In other words, the  $S_i$  and  $T_j$  are given by

$$(21) \quad S_1 = \frac{1}{2}(a + d + b + c) \quad S_2 = \frac{1}{2}(a + d + e + f) \quad S_3 = \frac{1}{2}(b + c + e + f)$$

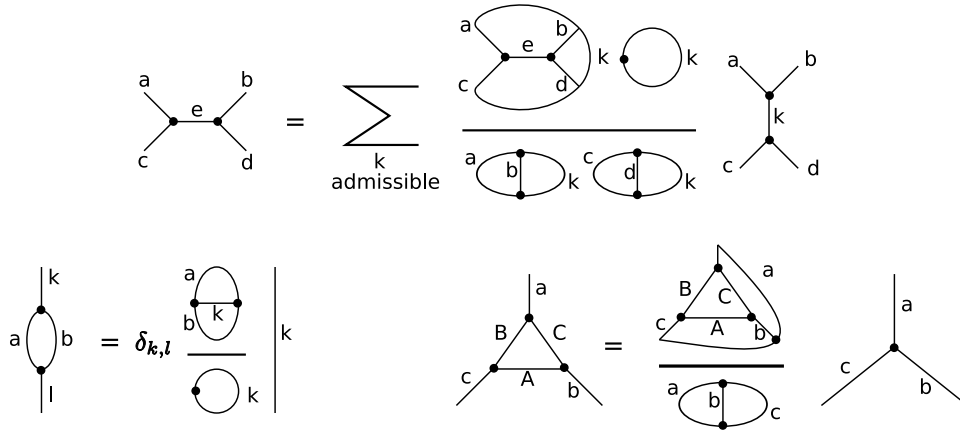
$$(22) \quad T_1 = \frac{1}{2}(a + b + e) \quad T_2 = \frac{1}{2}(a + c + f) \quad T_3 = \frac{1}{2}(c + d + e) \quad T_4 = \frac{1}{2}(b + d + f).$$

(b) Let  $(\Theta, \gamma)$  denote the  $\Theta$  spin network of Figure 1 admissibly colored by  $\gamma = (a, b, c)$ . Then we have

$$(23) \quad \langle \Theta, \gamma \rangle = (-1)^{\frac{a+b+c}{2}} \left( \frac{a+b+c}{2} + 1 \right) \left( \frac{a+b+c}{2}, \frac{-a+b+c}{2}, \frac{a-b+c}{2}, \frac{a+b-c}{2} \right).$$

Note that the value of a  $n$ -colored unknot is by Lemmas 6.1 and 6.3 equal to  $(-1)^n(n+1)$ .

Recoupling is a way to modify a spin network locally, while preserving its evaluation. This is done as in the topmost picture in Figure 4. This formula is called the recoupling formula and follows from the recoupling formula in [KL], using our conventions. The other two pictures in the figure show two important special cases of the recoupling formula, the bubble formula and the triangle formula.



**Figure 4.** The recoupling formula (top), the bubble formula (left) and the triangle formula (right). The sum is over all  $k$  for which the network is admissible, and  $\delta_{k,l}$  is the *Kronecker delta function*.

The bubble formula shown on the left of Figure 4 serves to remove all bigon faces. Likewise the triangle formula can be used to remove triangles. To see why the recoupling formula suffices to write any spin network as a multi sum of products of  $6j$ -symbols divided by thetas we argue as follows. Applying the recoupling formula to a curve in the graph reduces its length by one. Keep going until you get a multiple edge which can then be removed by the bubble formula.

Although the triangle formula follows quickly from the bubble formula and the recoupling formula it is important enough to state on its own. This formula shows that the evaluation of the class of *triangular networks* is especially simple. The triangular networks, mentioned in Remark 3.6 are the planar graphs that can be obtained from the tetrahedron by repeatedly replacing a vertex by a triangle. By the triangle formula any triangular network is simply a product of  $6j$ -symbols divided by thetas.

To illustrate how recoupling theory works, let us evaluate the regular  $s$ -sided drum and  $K_{3,3}$ . Consider the  $s$ -sided drum network as shown in Equation (24) (for  $s = 5$ ) where every edge is colored by the integer  $n$ . In the figure we have left out most of the labels  $n$  for clarity. By convention unlabeled edges are colored by  $n$ . Performing the recoupling move on every inward pointing edge we transform the drum into a string of bubbles that is readily evaluated.

$$(24) \quad \begin{array}{c} \text{Diagram of a tetrahedron with } n \text{ edges} \end{array} = \sum_{k \text{ admissible}} \left( \frac{\begin{array}{c} \text{Diagram of a tetrahedron with } k \text{ edges} \\ \text{Diagram of a tetrahedron with } k \text{ edges} \end{array}}{\begin{array}{c} \text{Diagram of a tetrahedron with } k \text{ edges} \\ \text{Diagram of a tetrahedron with } k \text{ edges} \end{array}} \right)^s \begin{array}{c} \text{Diagram of a tetrahedron with } k \text{ edges} \end{array} = \sum_{k \text{ admissible}} \left( \frac{\begin{array}{c} \text{Diagram of a tetrahedron with } k \text{ edges} \\ \text{Diagram of a tetrahedron with } k \text{ edges} \end{array}}{\begin{array}{c} \text{Diagram of a tetrahedron with } k \text{ edges} \\ \text{Diagram of a tetrahedron with } k \text{ edges} \end{array}} \right)^s \begin{array}{c} \text{Diagram of a tetrahedron with } k \text{ edges} \end{array}$$

Observing that if  $n$  is odd the network is not admissible (and thus evaluates to zero), and denoting the tetrahedron and the theta with one edge colored by  $k$  and the others by  $n$  by  $S(n, k)$  and  $\theta(n, k)$  we conclude the following formula for the  $n$ -colored  $s$ -sided drum.

**Proposition 6.4.** *If  $n = 2N$  is even we have*

$$\langle \text{Drum}_s, 2N \rangle = \sum_{j=0}^{2N} (2j+1) \left( \frac{S(2N, 2j)}{\theta(2N, 2j)} \right)^s$$

and if  $n$  is odd we have  $\langle \text{Drum}_s, n \rangle = 0$ .

For small values of  $s$  the drum can be evaluated in a more straight forward way, thus providing some well known identities of  $6j$ -symbols. Namely when  $s = 1$  we get zero, when  $s = 2$  we find some thetas and when  $s = 3$  we have by the triangle formula a product of two  $6j$ -symbols thus giving a special case of the Biedenharn-Elliott identity [KL]. For  $s = 4$  we find a formula for the regular cube, we know of no easier evaluation in this case.

A similar computation for  $K_{3,3}$  cyclically ordered as a plane hexagon with its three diagonals gives the following.

**Proposition 6.5.** *If  $n = 2N$  is even we have*

$$\langle K_{3,3}, 2N \rangle = \sum_{j=0}^{2N} (-1)^j (2j+1) \left( \frac{S(2N, 2j)}{\theta(2N, 2j)} \right)^3$$

and if  $n$  is odd we have  $\langle K_{3,3}, n \rangle = 0$ .

Note the similarity between  $\text{Drum}_3$  and  $K_{3,3}$ . The only difference is the sign that comes up in the calculation when one needs to change the cyclic order. The extra sign makes  $\langle K_{3,3}, 2N \rangle = 0$  for all odd  $N$ . This is because changing the cyclic ordering at a vertex takes the graph into itself, while it produces a sign  $(-1)^N$  when all edges are colored  $2N$ .

## 7. $G$ -FUNCTIONS AND EXISTENCE OF ASYMPTOTIC EXPANSIONS

**7.1. Existence of asymptotic expansions.** In this section we prove the existence of an asymptotic expansion for all spin networks using the theory of  $G$ -functions and the combinatorial fact that any spin network can be evaluated as a multi-dimensional sum of a balanced hypergeometric term.

**Definition 7.1.** An  $r$ -dimensional *balanced hypergeometric term*  $\mathfrak{t}_{n,k}$  (in short, *balanced term*) in variables  $(n, k)$ , where  $n \in \mathbb{N}$  and  $k = (k_1, \dots, k_r) \in \mathbb{Z}^r$ , is an expression of the form

$$(25) \quad \mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^r C_i^{r_i} \prod_{j=1}^J A_j(n, k)^{\epsilon_j}$$

where  $C_i$  are algebraic numbers for  $i = 0, \dots, r$ ,  $\epsilon_j = \pm 1$  for  $j = 1, \dots, J$ , and  $A_j$  are integral affine linear forms in  $(n, k)$  that satisfy the *balance condition*

$$(26) \quad \sum_{j=1}^J \epsilon_j A_j = \text{constant}$$

We assume that the convex polytope  $P_{\mathbf{t}}$  in  $\mathbb{R}^r$  defined by  $A_j(1, w) \geq 0$  for  $j = 1, \dots, J$  and  $w = (w_1, \dots, w_r)$  is compact.

**Lemma 7.2.** *For every spin network  $(\Gamma, \gamma)$  there exists a balanced term  $\mathbf{t}(n, k)$  with  $k = (k_1, \dots, k_r)$  such that*

$$(27) \quad \langle \Gamma, n\gamma \rangle = \sum_{k \in nP_{\mathbf{t}} \cap \mathbb{Z}^r} \mathbf{t}(n, k)$$

for every  $n \in \mathbb{N}$ .

*Proof.* Using the recoupling formulae from the previous paragraph we can write  $\langle \Gamma, \gamma \rangle$  as a multi-dimensional sum of products of  $6j$ -symbols,  $3j$ -symbols and unknots (i.e.,  $1j$ -symbols) with a denominator consisting of  $3j$ -symbols. It follows from Equations (20) and (23) that the  $6j$ -symbols (resp.  $3j$ -symbols) are balanced 1-dimensional (resp. 0-dimensional) sums, thus the ratio of the product of the  $3j$ -symbols by the product of the  $3j$ -symbols is a balanced multi-dimensional sum. The unknots can be written as  $(-1)^k (k+1)!/k!$  and are therefore balanced as well. It is easy to check that admissibility guarantees that the multi-dimensional sum has finite range.  $\square$

Beware that the term  $\mathbf{t}(n, k)$  constructed in the proof of lemma 7.2 is neither unique nor canonical in any sense. Lemma 7.2 and the following result implies Theorem 1.

**Theorem 7.** [Ga4, Thm.2] (a) If  $(a_n)$  is a multi-dimensional sum of a balanced term, then the generating series

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a  $G$ -function.

(b) If  $F(z)$  is a  $G$ -function, its Taylor series at  $z = 0$  is a sequence of Nilsson type.

*Remark 7.3.* This argument also implies that the sequences  $\langle \Gamma, n\gamma \rangle^U$  and  $\langle \Gamma, n\gamma \rangle^P$  are also asymptotic to a rational power of  $n!$  times a sequence of Nilsson type. This follows from the fact that the above normalizations of the spin networks differ from Definition 1.2 by a ratio of factorials of linear forms in  $n$ , which can be asymptotically expanded using Stirling's formula.

Let us end this section with a lemma which shows that diagonals of rational functions are  $G$ -functions that come from geometry. This is relevant to spin networks since we know from Theorem 6 that the generating function  $F_{\Gamma, \gamma}$  for the evaluations of a spin network is a diagonal of a rational function.

Fix a power series  $f(x_1, \dots, x_r) \in \mathbb{Q}[[x_1, \dots, x_r]]$  convergent at the origin and an exponent  $I = (i_1, \dots, i_r) \in \mathbb{N}_+^r$  and consider the diagonal  $(\Delta_I f)(z) \in \mathbb{Q}[[z]]$  as in Definition 4.5. Let  $\mathcal{C}$  denote a small  $r$ -dimensional torus around the origin. Then we have the following.

**Lemma 7.4.** *With the above assumptions, we have*

$$(28) \quad (\Delta_I f)(z) = \frac{1}{(2\pi i)^r} \int_{\mathcal{C}} \frac{f(x_1, \dots, x_r)}{x_1 \dots x_r - z} dx_1 \wedge \dots \wedge dx_r.$$

*Proof.* With the notation of Definition 16, an application of Cauchy's theorem gives for every natural number  $n$

$$[x^{nI}](f) = \frac{1}{(2\pi i)^r} \int_{\mathcal{C}} \frac{f(x_1, \dots, x_r)}{(x_1 \dots x_r)^{n+1}} dx_1 \wedge \dots \wedge dx_r.$$

Summing up for  $n$  and interchanging summation and integration concludes the proof.  $\square$

If in addition  $f(x_1, \dots, x_r)$  is a rational function, then the singularities of the analytic continuation of the right hand-side of (28) can be analyzed by deforming the integration cycle  $\mathcal{C}$  and studying the corresponding variation of Mixed Hodge Structure as in [BK]. Such  $G$ -functions come from geometry; see [An, BK].

## 8. COMPUTATION OF ASYMPTOTIC EXPANSIONS USING BOREL TRANSFORM

The results of Section 7 are powerful but non-constructive. In this section we will use constructive results from the theory of Borel transform, known in the case of balanced terms of rank 1. These results appear in [CG1, CG2, Ga4], and in the present paper we will quote them without proofs. Let us specialize to 1-dimensional balanced terms  $t_{n,k}$ , that is expressions of the form

$$(29) \quad t_{n,k} = C_0^n C_1^k \prod_{j=1}^J A_j(n, k)!^{\epsilon_j}$$

satisfying the balancing condition (26). The relevant example of these is the  $6j$ -symbol, Equation (20). Fixing a 1-dimensional balanced term  $t_{n,k}$ , summation over  $k$  (it is a finite sum) gives rise to a sequence

$$(30) \quad a_n = \sum_{k \in \mathbb{Z}} t_{n,k}$$

To describe and compute the asymptotics of such sequences we need the following notation. For affine linear forms we use the notation  $A(n, k) = v_0(A)n + v_1(A)k + v(A)$  and  $A(u) = v_0(A) + v_1(A)u$ . Next we define the multivalued analytic functions

$$(31) \quad V(u) = \log C_0 + u \log C_1 + \sum_{j=1}^J \epsilon_j A_j(u) \log(A_j(u))$$

$$(32) \quad W(u) = \prod_{j=1}^J A_j(u)^{\epsilon_j/2 + \epsilon_j v(A_j)}$$

$V$  and  $W$  are analytic functions on the open interval  $(m, M)$  (which is determined by  $A_j, j = 1, \dots, J$ ) and they have single-valued analytic continuation in the *doubly cut* complex plane  $\mathbb{C} \setminus ((-\infty, m] \cup [M, \infty))$ .  $V$  and  $W$  can be further analytically continued as multivalued functions in the complex plane minus a finite set of points  $\mathbb{C} \setminus \{u \prod_j A_j(u) = 0\}$ .

Define the constants

$$(33) \quad \mu = \frac{1}{2} \sum_{j=1}^J \epsilon_j, \quad \nu = \sum_{j=1}^J \epsilon_j v(A_j),$$

The set of critical points of  $V$  satisfies the Variational Equation

$$(34) \quad C_1 \prod_{j=1}^J A_j(u)^{\epsilon_j v_1(A_j)} = 1$$

If  $u$  is a critical point of  $V$ , then  $e^{V(u)} = \lambda(u)$  where

$$(35) \quad \lambda(u) = C_0 \prod_{j: A_j(u) \neq 0} A_j(u)^{\epsilon_j v_0(A_j)}$$

Let  $K$  denote the number field generated by the solutions of the Variational Equation.

Suppose that  $u$  is a critical point of order  $m$  for  $V$ , that is  $V^{(j)}(u) = 0$  for  $j = 1, \dots, m$  and  $V^{(m+1)}(u) \neq 0$  and  $W$  is non-vanishing at  $u$ , i.e.,  $W(u) \neq 0$ . Let

$$(36) \quad c_m = 2\Gamma\left(\frac{m+2}{m+1}\right) (m+1)!^{\frac{1}{m+1}}.$$

and let

$$(37) \quad \alpha(u) = \mu + \nu + 1 - \frac{1}{m+1}$$

and

$$(38) \quad S(u) = (2\pi)^\mu c_m \frac{W(u)}{(-V^{(m+1)}(u))^{\frac{1}{m+1}}}$$

We will require a modification of the above definition of the Stokes constant when  $u$  lies in one of the cuts  $(-\infty, m] \cup [M, \infty)$ . In that case, we define

$$(39) \quad S(u) = (2\pi)^\mu c_m \frac{|W(u)|}{(|V^{(m+1)}(u)|)^{\frac{1}{m+1}}}$$

**Theorem 8.** *With the above assumptions we have*

$$(40) \quad a_n \sim \sum_u \lambda(u)^n n^{\alpha_u} S(u) h_u \left( \frac{1}{n} \right)$$

where  $h_u \in K[[x]]$  is Gevrey-1 and  $u$  runs through a subset of solutions of the Variational Equation. The selected subset can be described algorithmically and in case of the  $6j$ -symbols it is discussed below.

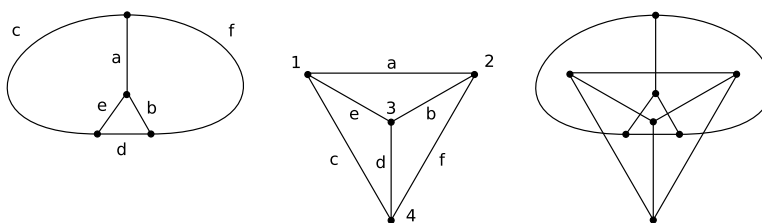
## 9. ASYMPTOTICS OF THE $6j$ -SYMBOL

The purpose of this section is to prove and explain Theorem 4 on the explicit expansions of the  $6j$ -symbols. Since the asymptotics are phrased in terms of geometric properties of the tetrahedron dual to the spin network, we start with a section on the geometry of the tetrahedron. Next we apply Theorem 8 to find the asymptotic expansion and finally we work out the details in each of the geometric cases.

**9.1. The geometry of the tetrahedron.** The asymptotics of the  $6j$ -symbols turn out to be intimately related to the geometry of the planar dual tetrahedron. From now on, we will assume that  $\gamma$  is non-degenerate, i.e. that the faces of  $(\nabla, \gamma)$  are two dimensional triangles.

Recall that the  $6j$ -symbol is a tetrahedral spin network  $(\triangle, \gamma)$  admissibly labeled as in Figure 1 with  $\gamma = (a, b, c, d, e, f)$ . Its dual tetrahedron  $(\nabla, \gamma)$  is also labeled by  $\gamma$ . The tetrahedron and its planar dual, together with an ordering of the vertices and a coloring of the edges of the dual is depicted in Figure 5. When a more systematic notation for the edge labels is needed we will denote them by  $d_{ij}$  (or  $ij$  if there cannot arise any confusion) it follows that

$$(a, b, c, d, e, f) = (d_{12}, d_{23}, d_{14}, d_{34}, d_{13}, d_{24})$$



**Figure 5.** The tetrahedral spin network and its planar dual with a vertex and edge labeling.

Recall from Section 3.1 that we interpret the labels of the dual tetrahedron  $\nabla$  as edge lengths in a suitable flat geometry. One may now ask for a condition that allows one to realize  $(\nabla, \gamma)$  in a flat metric space such that the edge lengths equal the edge labels. Labeling the vertices of  $(\nabla, \gamma)$  as in Figure 5 we can formulate such a condition in terms of the Cayley-Menger determinant. This is a homogeneous polynomial of degree 3 in the six variables  $a^2, \dots, f^2$ . We give a definition of the determinant following [Ko1].

**Definition 9.1.** Given numbers  $d_{ij}$  define the (modified) *Cayley-Menger* matrix  $(C_{ij})$  by  $C_{ij} = 1 - d_{ij}^2/2$  for  $i, j \geq 1$  and  $C_{ij} = \text{sgn}(i - j)$  for when  $i = 0$  or  $j = 0$ . In terms of the coloring  $\gamma = (a, b, c, d, e, f)$  of a tetrahedron, we have

$$(41) \quad C = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 - a^2/2 & 1 - e^2/2 & 1 - c^2/2 \\ 1 & 1 - a^2/2 & 1 & 1 - b^2/2 & 1 - f^2/2 \\ 1 & 1 - e^2/2 & 1 - b^2/2 & 1 & 1 - d^2/2 \\ 1 & 1 - c^2/2 & 1 - f^2/2 & 1 - d^2/2 & 1 \end{pmatrix}.$$

We can now state the criterion for the realizability of  $(\nabla, \gamma)$ . For a proof see [Bl] or [K01].

**Proposition 9.2.** *Consider a non-degenerate tetrahedron  $(\nabla, \gamma)$ . The sign of the Cayley-Menger determinant determines in what space the tetrahedron can be realized such that the edge labels equal the edge lengths*

- If  $\det(C) > 0$  then the tetrahedron is realized in Euclidean space  $\mathbb{R}^3$ .
- If  $\det(C) = 0$  then the tetrahedron is realized in the Euclidean plane  $\mathbb{R}^2$ .
- If  $\det(C) < 0$  then the tetrahedron is realized in Minkowski space  $\mathbb{R}^{2,1}$ .

In each case the volume of the tetrahedron is given by  $\text{Vol} = \frac{1}{6} \sqrt{|\det(C)|}$ .

The 6 dimensional space of non-degenerate tetrahedra thus consists of regions of Minkowskian and regions of Euclidean tetrahedra. It turns out to be a cone that is made up from one connected component of three dimensional Euclidean tetrahedra and two connected components of Minkowskian tetrahedra. The three dimensional Euclidean and Minkowskian tetrahedra are separated by Plane tetrahedra. The Plane tetrahedra also form two connected components, representatives of which are depicted in Figure 3. The tetrahedra in the Plane component that look like a triangle with an interior point are called *triangular* and the Plane tetrahedra from that other component that look like a quadrangle together with its diagonals are called *quadrangular*. We use the same names for the corresponding Minkowskian components. An integer representative of the triangular Plane tetrahedra is not easy to find as the smallest example is  $(37, 37, 13, 13, 24, 30)$ .

We finish this section with a short discussion of the dihedral angles of a tetrahedron realized in either of the three above spaces. The cosine and sine of these angles can be expressed in terms of certain minors of the Cayley-Menger matrix. Define the *adjugate matrix*  $\text{ad}(C)$  whose  $ij$  entry is  $(-1)^{i+j}$  times the determinant of the matrix obtained from  $C$  by deleting the  $i$ -th row and the  $j$ -th column. Define  $C_{iijj}$  to be the matrix obtained from  $C$  by deleting both the  $i$ -th row and column and the  $j$ -th row and column.

The Law of Sines and the Law of Cosines are well-known formulae for a triangle in the Euclidean plane. The next Lemma is the *Law of Sines* and the *Law of Cosines* for a tetrahedron in all three flat geometries.

**Lemma 9.3.** *Let  $\theta_{kl}$  be the exterior dihedral angle at the opposite edge  $ij$ . The following formula is valid for all non-degenerate tetrahedra*

$$\exp(\theta_{kl}i) = \frac{\text{ad}(C)_{kl} + \sqrt{-\det(C) \det(C_{kkll})}}{\sqrt{\text{ad}(C)_{kk} \text{ad}(C)_{ll}}}$$

*Proof.* See [K01] for a proof in the Euclidean and Plane case. The Minkowskian case is entirely analogous once one uses the right conventions on angles [SP]. Given a Minkowskian tetrahedron the angle between two outward pointing normal vectors has the form  $n\pi + i\omega$ , where  $n \in \mathbb{N}$  and  $\omega \in \mathbb{R}$ . Since the normal vectors are time-like they can either be in the same sheet or in different sheets of the unit hyperboloid. If they are in the same sheet then  $\omega$  is defined to be the length the hyperbolic line between them and  $n = 0$ . If they are in opposite sheets then  $-\omega$  is the length of the line between one and the reflection of the other in the horizontal plane and  $n = 1$ .  $\square$

**9.2. Asymptotics of the  $6j$ -symbol.** We now start to apply Theorem 8 to determine the asymptotics of the non-degenerate  $6j$ -symbol. In this section it is convenient to use the following normalization  $\langle \Gamma, \gamma \rangle^U$  for spin networks.

**Definition 9.4.** We define the *unitary normalization*  $\langle \Gamma, \gamma \rangle^U$  of a spin network evaluation  $(\Gamma, \gamma)$  to be

$$\langle \Gamma, \gamma \rangle^U = \frac{1}{\Theta(\gamma)} \langle \Gamma, \gamma \rangle$$

where

$$\Theta(\gamma) = \prod_{v \in V(\Gamma)} \sqrt{|\langle \Theta, a_v, b_v, c_v \rangle|}$$

and  $a_v, b_v, c_v$  are the colors of the edges at vertex  $v$ .

As it turns out, with this normalization the asymptotics of the  $6j$ -symbols matches the corresponding geometry well. Since the theta is a quotient of factorials one can use Stirling's formula [O] to find its asymptotics and hence one can see how the new choice in normalization effects the asymptotic expansion. For later use we record the asymptotic expansion of the normalization factor in the case of a  $6j$ -symbol.

Let us define

$$\lambda_{abc} = \frac{(-a+b+c)^{(-a+b+c)/4} (a-b+c)^{(a-b+c)/4} (a+b-c)^{(a+b-c)/4}}{(a+b+c)^{(a+b+c)/4}}$$

$$s_{abc} = \frac{((-a+b+c)(a-b+c)(a+b-c)(a+b+c))^{1/4}}{a+b+c}$$

**Lemma 9.5.** *Let  $\gamma$  be an admissible coloring of a tetrahedron. Then we have*

$$\Theta(n\gamma)^{-1} \sim (2\pi)^2 s_{abe} s_{acf} s_{ced} s_{bdf} (\lambda_{abe} \lambda_{acf} \lambda_{ced} \lambda_{bdf})^n \left( \sum_{l=0}^{\infty} \frac{\mu_l}{n^l} \right)$$

where  $\mu_0 = 1$  and  $\mu_l \in \mathbb{Q}(\gamma)$  for all  $l$ .

Thus, the asymptotics of  $\langle \triangle, \gamma \rangle^U$ , is reduced to the study the asymptotics of  $\langle \triangle, \gamma \rangle$ . Since this is a 1-dimensional sum of a balanced term given by Equation (20), we can apply Theorem 8 to find its asymptotic expansion. Equation (20) implies that the balanced term is given by

$$(42) \quad t_{n,k} = \frac{(-1)^k (k+1)!}{\prod_i (nS_i - k)! \prod_j (k - nT_j)!}$$

With the notation of Theorem 8 we have  $J = 8$ , and the linear forms  $A_j(n, k)$  are given by

$$A_1(n, k) = k + 1, \quad A_2(n, k) = nS_1 - k, \quad A_3(n, k) = nS_2 - k, \quad A_4(n, k) = nS_3 - k,$$

$$A_5(n, k) = k - nT_1, \quad A_6(n, k) = k - nT_2, \quad A_7(n, k) = k - nT_3, \quad A_8(n, k) = k - nT_4$$

and  $\epsilon_1 = 1, \epsilon_2 = \dots \epsilon_8 = -1$ . It follows that

$$(43) \quad \nu = 1, \quad \mu = -3.$$

The functions  $V$  and  $W$  from Theorem 8 in this case are

$$(44) \quad V(u) = u \log(-1) + u \log u - \sum (S_i - u) \log(S_i - u) - \sum (u - T_j) \log(u - T_j)$$

$$(45) \quad W(u) = u^{3/2} (\prod (S_i - u) \prod (u - T_j))^{-1/2}.$$

$V$  and  $W$  are analytic functions on the open interval

$$(m, M) = (\max\{0, T_j\}, \min\{S_i\})$$

and they have single-valued analytic continuation in the doubly cut complex plane  $\mathbb{C} \setminus ((-\infty, \max\{0, T_j\}] \cup [\min\{S_i\}, \infty))$ .  $V$  and  $W$  can be further analytically continued as multivalued functions in  $\mathbb{C} \setminus \{0, S_i, T_j\}$ .

The possible critical values of  $V(u)$  that determine the growth rate are given by

$$(46) \quad \lambda_v = \frac{\prod_{j=1}^4 (v - T_j)^{T_j}}{\prod_{i=1}^3 (S_i - v)^{S_i}}$$

here  $v$  is a solution of the variational equation which is the equation

$$(47) \quad -u \prod (S_i - u) \prod (u - T_j)^{-1} = 1$$

At first sight, the above equation appears to be a quartic equation for  $u$ . However, the coefficients of  $u^4$  and  $u^3$  is zero because  $\sum S_j = \sum T_i$ . Thus, Equation (47) is quadratic in  $u$ , with coefficients which are polynomial in  $a, \dots, f$ . It is convenient to introduce a fourth number  $S_4 = 0$ . We can then write the Variational Equation (47) in the form

$$(48) \quad E(u) = -\prod (S_i - u) + \prod (u - T_j) = Au^2 + Bu + C' = 0$$

where  $A$ ,  $B$  and  $C'$  are homogeneous polynomials in  $\gamma$  of degrees 2, 3 and 4 respectively. A calculation gives that

$$(49) \quad A = \frac{1}{2}(ad + bc + ef)$$

$$(50) \quad B = -\frac{1}{4}(bc(b+c) + ad(a+d) + ef(e+f) + abc + abd + acd + bcd + abc + bce + ade + cde + acf + bcf + adf + bdf + aef + bef + cef + def)$$

The next Lemma identifies the number field generated by the Variational Equation (48) in terms of the Cayley-Menger determinant.

**Lemma 9.6.** *Let  $D$  be the discriminant of the Variational Equation (47). Then we have*

$$(51) \quad D = -\det(C)/4$$

where  $C$  is the Cayley-Menger matrix. The number field  $K$  generated by the solutions to the Variational Equation (48) is given by

$$(52) \quad K = Q(\sqrt{-\det(C)}).$$

*Proof.* The proof follows from an explicit calculation, keeping in mind that  $D$  and  $\det(C)$  are homogeneous polynomials in  $\gamma$  of degree 6.  $\square$

The next Lemma determines the order of vanishing of  $V$  at the solutions of Equation (47).

**Lemma 9.7.** *Let  $D$  be the discriminant of the Variational Equation (47).*

- (a) *The tetrahedron is degenerate iff there is a solution to the variational equation that equals  $S_i$ , or  $T_j$  for some  $i, j$ .*
- (b) *Assume the tetrahedron is non-degenerate. The two solutions  $v_+, v_-$  to the variational equation satisfy*

$$-\prod (v_{\pm} - T_j) V''(v_{\pm}) = \pm \sqrt{D}$$

- (c) *If in addition  $D = 0$  then the two solutions coincide,  $V''(v_{\pm}) = 0$  and we have*

$$\prod (v - T_j) V'''(v) = -2A$$

where  $A$  is given in (49).

- (d) *In the Euclidean or Plane case,  $v_{\pm}$  does not lie in the double cut  $(-\infty, m] \cup [M, \infty)$ . In the Minkowskian case,  $v_{\pm}$  are real and exactly one of them (call it  $v$ ) lies in the double cut  $(-\infty, m] \cup [M, \infty)$ .*

*Proof.* In the proof it is convenient to define  $S_4 = 0$  so that all indices run from 1 to 4. This convention will only be used in this proof. (a) If the tetrahedron is degenerate then there are  $i, j$  such that  $S_i = T_j$ . Plugging in this value for  $u$  will set both terms in the equation to zero as required. On the other hand if  $v = T_j$  is a solution to the variational equation then  $\prod_i (S_i - T_j) = 0$ , so that  $T_j = S_i$  for some  $i$  and likewise when  $v = S_i$ .

(b) Note that  $A = (ad + bc + ef)/2 > 0$  since all edge labels are positive so that a solution always exists. By (a) we know that the solutions to the variational equation are not equal to  $S_i$  or  $T_j$ .

By differentiating  $V$  we find that  $-V''(v_{\pm}) = \sum \frac{1}{S_i - v_{\pm}} + \sum \frac{1}{v_{\pm} - T_j}$ . Using the variational equation  $\prod(S_i - u) = \prod(u - T_j)$  we can write

$$(53) \quad -\prod(v_{\pm} - T_j)V''(v_{\pm}) = \prod(S_i - v_{\pm}) \sum \frac{1}{S_j - v_{\pm}} + \prod(v_{\pm} - T_i) \sum \frac{1}{v_{\pm} - T_j}$$

We see that  $-\prod(v_{\pm} - T_j)V''(v_{\pm}) = E'(v_{\pm})$ . Now choose the solutions such that  $v_{\pm} = (-B \pm \sqrt{D})/2A$  so that  $E'(v_{\pm}) = 2Av_{\pm} + B = \pm\sqrt{D}$ .

(c) Since  $D = 0$  the solutions  $v_{\pm}$  coincide and will be written as  $v$ . Differentiating we find  $V'''(v) = -\prod(S_i - v)^{-2} + \prod(v - T_j)^{-2}$ . On the other hand  $D = 0$  and we assumed that our tetrahedron is non-degenerate, so by part (a)  $V''(v) = 0$ . Therefore

$$\frac{1}{(S_i - v)^2} + \sum_{j \neq i} \frac{1}{(S_i - v)(S_j - v)} + \sum_k \frac{1}{(S_i - v)(v - T_k)} = \frac{V''(v)}{S_i - v} = 0$$

Adding these to similar equations for  $\frac{V''(v)}{v - T_j}$  implies

$$V'''(v) = 2 \sum_{i \neq j} \frac{1}{(S_i - v)(S_j - v)} - \frac{1}{(v - T_i)(v - T_j)}$$

Multiplying through by the variational equation  $\prod(v - T_j) = \prod(S_i - v)$  we get

$$\prod(v - T_j)V'''(v) = 2 \sum_{i \neq j} (S_i - v)(S_j - v) - (v - T_i)(v - T_j) = -2A,$$

where  $A$  is given in Equation (49).

(d) The Euclidean or Plane case follows from (b). For the Minkowskian case, there are connected components (types) of Minkowskian tetrahedra: the rectangular type where  $T_j < v_{\pm}$  and the *triangular type* where we have  $T_1, T_2, T_3 < v_{\pm} < T_4$  where  $T_4$  is the distinguished triangle of the triangular tetrahedron. This fact is in turn proven by continuity. It is true by explicit computation (using the WZ method of Section 10) for the two examples

$$(3, 4, 4, 3, 5, 5), \quad (37, 37, 13, 13, 24, 30)$$

of rectangular type and triangular type respectively. The result follows by continuity for all Minkowskian nondegenerate tetrahedra, since  $v_{\pm} = T_j$  only for degenerate tetrahedra.  $\square$

From now on we will always assume that our  $6j$ -symbols are non-degenerate. Using Lemma 9.5 and Theorem 8, it follows that the possible growth rates in the asymptotic expansion of  $\langle \triangle, \gamma \rangle^U$  are given by

$$\Lambda_{\pm} = \lambda_{abe} \lambda_{acf} \lambda_{ced} \lambda_{bdf} \lambda_{v_{\pm}}$$

We close this subsection with a lemma connecting these growth rates to the dihedral angles in the tetrahedron.

**Lemma 9.8.** *Let  $\theta_a$  be the dihedral angle corresponding to the edge labeled  $a$  as introduced in Section 9.1.*

(a) *The growth rate  $\Lambda_{\pm}$  satisfies*

$$(54) \quad \Lambda_{\pm} = e^{\pm i \sum_a \theta_a \frac{a}{2}}$$

(b) *Moreover,*

$$(55) \quad (s_{abe} s_{acf} s_{cde} s_{bdf} W_t(v_{\pm}))^2 = \frac{-e^{\pm i \sum_a \theta_a a}}{\prod_j (v_{\pm} - T_j)}$$

*Proof.* (a) The strategy is to collect equal powers in  $\Lambda_{\pm}$ . It follows that we can write  $\Lambda_{\pm}$  as a product

$$\Lambda_{\pm} = \prod_{i < j} (h_{ij\pm})^{d_{ij}/2}$$

where we define

$$(56) \quad h_{ij\pm} = \frac{(v_{\pm} - (ij + ik + jk)/2)(v_{\pm} - (ij + il + jl)/2)(ij - ik + jk)(ij + ik - jk)(ij - il + jl)(ij + il - jl)}{((ij + kl + ik + jl)/2 - v_{\pm})((ij + kl + jk + il)/2 - v_{\pm})\sqrt{K_l K_k}}$$

and

$$K_l = (-ij + ik + jk)(ij - ik + jk)(ij + ik - jk)(ij + ik + jk)$$

Now we will show that  $h_{ij\pm} = e^{\pm\theta_{ki}i}$ . Since  $\det(C_{kkll}) = d_{ij}^2$  and  $4\text{ad}_l(C) = K_l$  we can clear denominators in the expression for  $e^{\pm\theta_{ki}i}$  from Lemma 9.3 and the expression for  $h_{ij\pm}$ . Equality of the numerators is then shown by direct calculation using Equation (51). Equation (54) follows.

(b) To show Equation (55), first note that  $s_{ij} s_{ik} s_{jk} = \frac{(K_l)^{1/4}}{ij+ik+jk}$  and hence

$$(s_{abe}s_{acf}s_{cde}s_{bdf}W_t(v_{\pm}))^2 = \frac{v_{\pm}^3 \sqrt{K_1 K_2 K_3 K_4}}{\prod(S_i - v_{\pm}) \prod(v_{\pm} - T_j) (T_1 T_2 T_3 T_4)^2}$$

In the product of the  $h_{ij}$  we can also collect like terms together in the form  $K_j$  in the denominator. We get

$$\prod_{i < j} h_{ij} = \frac{(K_1 K_2 K_3 K_4)^{3/2}}{(T_1 T_2 T_3 T_4)^2 K_1 K_2 K_3 K_4} \frac{\prod(v_{\pm} - T_j)^3}{\prod(S_i - v_{\pm})^4}$$

Using the variational equation and  $h_{ij\pm} = e^{\pm\theta_{ki}i}$  proves part (b).  $\square$

Lemmas 9.8 and 9.7 prove Lemma 3.2 regarding the properties of the growth rates.

Combining our results, we can finally prove Theorem 4 by reading off the numerical invariants from Theorem 8. First of all the growth rates  $\Lambda_{\pm}$  are identified with the dihedral angles by Lemma 9.8. In the Euclidean and the Plane case both rates contribute, whereas in the Minkowskian case we include only the contribution of  $\Lambda$ , corresponding to the distinguished solution  $v$  of the Variational Equation that lies in the cut; see part (d) of Lemma 9.7. In the terminology of Section 9.1 and Lemma 9.7 we have  $\Lambda = \Lambda_+$  in the triangular component and  $\Lambda = \Lambda_-$  in the quadrangular component, and always  $|\Lambda| < 1$ .

Next the exponent in the leading power of  $n$  is given by Equation (37) and the order of vanishing is determined by Lemma 9.7. We have  $m = 1$  in the Euclidean and Minkowskian cases and  $m = 2$  in the Plane cases.

Lemma 9.6 identifies the number field generated by the solutions to the Variational Equation with  $K$  from Definition 3.1.

Finally we consider the Stokes constants in all three cases. The Stokes constant is computed from Theorem 8 using Lemmas 9.8 and 9.7. In the Euclidean case we have to refine Lemma 9.8 (b) to choose the right square root of the left hand side. This is done by considering the simple example of the regular tetrahedron and extending by continuity. We thus find

$$S_{\pm} = (2\pi)^{-1/2} s_{abe}s_{acf}s_{cde}s_{bdf} \frac{W_t(v_{\pm})}{\sqrt{V_t''(v_{\pm})}} = (2\pi)^{-1/2} \frac{\pm i \prod_{\alpha} e^{\pm\theta_{\alpha}i/2}}{\sqrt{-\prod(v_{\pm} - T_j) V''(v_{\pm})}}$$

Now applying part (b) of Lemma 9.7 and Lemma 9.6, we can rewrite the above as

$$S_{\pm} = (2\pi)^{-1/2} \frac{\pm i e^{\pm i \sum_{\alpha} \theta_{\alpha}/2}}{\sqrt{\pm i \sqrt{|\det(C)|}/2}} = \frac{e^{\pm i (\sum_{\alpha} \theta_{\alpha}/2 + \pi/4)}}{\sqrt{6\pi \text{Vol}}}$$

In the Minkowskian case we have to choose the Stokes constant corresponding to  $\Lambda$ . This is computed as in the Euclidean case except that the solutions to the variational equation lie in the cut. We therefore use Equation (39) so that we take absolute values everywhere. This explains the sign in (9). It makes the Stokes constant positive by canceling the real part of the Minkowskian angle sum, which is  $3\pi$  in the triangular case and  $4\pi$  in the quadrangular case.

Finally in the Plane case we have by Lemma 9.7 that  $v \neq T_j$  and we know that  $V''(v) = 0$ . Furthermore  $c_2 = \Gamma(4/3)6^{1/3}$ , so

$$S = \frac{\Gamma(4/3)6^{1/3}}{\pi} s_{abc} s_{acf} s_{ced} s_{bdf} \frac{|W(v)|}{|V'''(v)|^{1/3}} = \frac{\Gamma(4/3)6^{1/3} |e^{i \sum_a \theta_a/2}|}{\pi \prod_j (v - T_j)^{1/2} |V'''(v)|^{1/3}} = \frac{\Gamma(4/3)6^{1/3}}{\pi \prod_j (v - T_j)^{1/6} (2A)^{1/3}}$$

where we used part (c) of Lemma 9.7 and the fact that the dihedral angles are real. Since the tetrahedron is Plane, we have  $\sqrt{D} = 0$ , so  $v = -B/2A$  with the notation of (48). Therefore we can take out the  $2A$  terms to get

$$S = \frac{\Gamma(4/3)(12A)^{1/3}}{\pi \prod |B + 2AT_j|^{1/6}}$$

Recall from Equations (22), (49) and (50) that  $T_j$ ,  $A$  and  $B$  are homogeneous polynomials of  $\gamma$ . A calculation shows that  $B_j := B + 2AT_j$  is a homogeneous polynomial of degree 3. It is unclear to us what is the geometric interpretation of the rational function  $A^2 / \prod_{j=1}^4 B_j \in \mathbb{Q}(\gamma)$ .

## 10. COMPUTATION OF ASYMPTOTIC EXPANSIONS USING THE WZ METHOD

In this section we will compute the asymptotic expansion of three representative  $6j$ -symbols using the WZ method. The main observation is that a  $6j$ -symbol is given by one-dimensional sum of a balanced hypergeometric term; see Equation (20) and Definition 7.1. The fundamental theorem of Wilf-Zeilberger implies that such sums are holonomic, i.e., they satisfy linear recursion relations with coefficients polynomials in  $n$ . The WZ method is constructive and works well for one-dimensional sums, implemented in several platforms, see [PaRi, PWZ, WZ1]. Given a recursion relation for the evaluation of the  $6j$ -symbols, together with the fact that it is a sequence of Nilsson type, allows us to compute explicitly its asymptotic expansion.

The three examples we consider are the simplest representatives of the three geometric types of  $6j$ -symbols. Let

$$\gamma_{\text{Euclidean}} = (2, 2, 2, 2, 2, 2), \quad \gamma_{\text{Plane}} = (3, 4, 4, 3, 5, 5), \quad \gamma_{\text{Minkowskian}} = (4, 4, 4, 4, 6, 6).$$

and consider the sequences  $(a_n)$ ,  $(b_n)$  and  $(c_n)$  defined by

$$\begin{aligned} a_n &:= \langle \triangle, n \gamma_{\text{Euclidean}} \rangle^U = \frac{n!^6}{(3n+1)!^2} \sum_{k=3n}^{4n} \frac{(-1)^k (k+1)!}{(k-3n)!^4 (4n-k)!^3} \\ b_n &:= \langle \triangle, n \gamma_{\text{Plane}} \rangle^U = \frac{n!^2 (2n)!^2 (3n)!^2}{(6n+1)!^2} \sum_{k=6n}^{7n} \frac{(-1)^k (k+1)!}{(k-6n)!^4 (7n-k)! (8n-k)! (9n-k)!} \\ c_n &:= \langle \triangle, n \gamma_{\text{Minkowskian}} \rangle^U = \frac{n!^2 (3n)!^4}{(7n+1)!^2} \sum_{k=7n}^{8n} \frac{(-1)^k (k+1)!}{(k-7n)!^4 (8n-k)! (10n-k)!^2} \end{aligned}$$

The command

```
<< zb.m
```

loads the package of [PaRi] into Mathematica. The command

```
teucl[n_, k_] := n!^6 / (3n+1)!^2 (-1)^k (k+1)! / ((4n-k)!^3 (k-3n)!^4)
```

defines the summand of the sequence  $(a_n)$ , and the command

```
zb[teucl[n, k], {k, 3n, 4n}, n, 2]
```

computes the following second order linear recursion relation for the sequence  $(a_n)$

$$\begin{aligned} -9(1+n)(2+3n)^2(4+3n)^2(451+460n+115n^2)a[n] + \\ (3+2n)(319212+1427658n+2578232n^2+2423109n^3+1255139n^4+340515n^5+37835n^6)a[1+n] - \\ 9(2+n)(5+3n)^2(7+3n)^2(106+230n+115n^2)a[2+n] = 0 \end{aligned}$$

This linear recursion has two formal power series solutions of the form

$$\begin{aligned}
a_{\pm,n} &= \frac{1}{n^{3/2}} \Lambda_{\pm}^n \left( 1 + \frac{-432 \pm 31i\sqrt{2}}{576n} + \frac{109847 \mp 22320i\sqrt{2}}{331776n^2} + \frac{-18649008 \pm 4914305i\sqrt{2}}{573308928n^3} \right. \\
&+ \frac{14721750481 \pm 45578388960i\sqrt{2}}{660451885056n^4} + \frac{-83614134803760 \pm 7532932167923i\sqrt{2}}{380420285792256n^5} \\
&+ \left. \frac{-31784729861796581 \mp 212040612888146640i\sqrt{2}}{657366253849018368n^6} + O\left(\frac{1}{n^7}\right) \right)
\end{aligned}$$

where

$$\Lambda_{\pm} = \frac{329 \mp 460i\sqrt{2}}{729}$$

are two complex numbers of absolute value 1. These growth rates  $\Lambda_{\pm}$  computed with the WZ method match those of Theorem 4. Indeed, the Theorem states that

$$\Lambda_{\pm} = e^{\mp i6 \arccos(1/3)}$$

An algebraic calculation confirms that

$$e^{\pm i6 \arccos(1/3)} = \frac{329 \pm 460i\sqrt{2}}{729}.$$

The coefficients of the formal power series  $a_{\pm,n}$  are in the number field  $K = Q(\sqrt{-2})$  and the Cayley-Menger determinant is  $\det(C) = 2^5$  confirming Lemma 9.6 and Theorem 4.

More is actually true. Namely, the sequence  $(a_n)$  generates two new sequences  $(\mu_{+,n})$  and  $(\mu_{-,n})$  defined by

$$a_{\pm,n} = \frac{1}{n^{3/2}} \Lambda_{\pm}^n \sum_{l=0}^{\infty} \frac{\mu_{\pm,l}}{n^l}$$

where  $\mu_{\pm,0} = 1$ . Each of the sequences  $(\mu_{\pm,n})$  are factorially divergent. However, the generating series  $\sum_{n=0}^{\infty} z^n \mu_{\pm,n+1}/n!$  are  $G$ -functions (as follows from [An]), and the sequences  $(\mu_{\pm,n+1}/n!)$  are of Nilsson type, with exponential growth rates  $\Lambda_{\pm} - \Lambda_{\mp}$ . The asymptotics of each sequence  $(\mu_{\pm,n+1}/n!)$  gives rise to finitely many new sequences, and so on. All those sequences span a finite dimensional vector space, canonically attached to the sequence  $(a_n)$ . This is an instance of resurgence, and is explained in detail in [GM, Sec.4].

The second order recursion relation for the Plane and the Minkowskian examples has lengthy coefficients, and leads to the following sequences  $(b_{\pm,n})$  and  $(c_{\pm,n})$

$$\begin{aligned}
b_{+,n} &= \frac{1}{n^{4/3}} \Lambda_+^n \left( 1 - \frac{1}{3n} + \frac{3713}{46656n^2} - \frac{25427}{2239488n^3} + \frac{9063361}{17414258688n^4} - \frac{109895165}{104485552128n^5} \right. \\
&\quad \left. + \frac{1927530983327}{2437438960041984n^6} + \dots \right) \\
b_{-,n} &= \frac{1}{n^{5/3}} \Lambda_-^n \left( 1 - \frac{37}{96n} + \frac{3883}{46656n^2} - \frac{13129}{4478976n^3} - \frac{5700973}{8707129344n^4} - \frac{14855978561}{3343537668096n^5} \right. \\
&\quad \left. + \frac{2862335448661}{2437438960041984n^6} + \dots \right) \\
c_{\pm,n} &= \frac{1}{n^{3/2}} \Lambda_{\pm}^n \left( 1 + \frac{336 \mp 1369\sqrt{2}}{4032n} + \frac{1769489 \mp 831792\sqrt{2}}{1806336n^2} + \frac{67925105712 \mp 66827896993\sqrt{2}}{21849440256n^3} \right. \\
&\quad + \frac{5075437500833257 \mp 2589265090380768\sqrt{2}}{176193886224384n^4} \\
&\quad + \frac{100978405759997442992 \mp 98904713360431641651\sqrt{2}}{552544027199668224n^5} \\
&\quad \left. + \frac{685103512739058526758457 \mp 349782631602887151717776\sqrt{2}}{247539724185451364352n^6} + \dots \right)
\end{aligned}$$

where in the Plane case we have

$$\Lambda_- = \Lambda_+ = -1, \quad K = \mathbb{Q}, \quad \det(C) = 0$$

and in the Minkowskian case we have

$$\begin{aligned}
\Lambda_+ &= \frac{696321931873 - 111529584108\sqrt{2}}{678223072849} = 0.794127\dots \\
\Lambda_- &= \frac{696321931873 + 111529584108\sqrt{2}}{678223072849} = 1.25924\dots \\
K &= \mathbb{Q}(\sqrt{2}), \quad \det(C) = -2^5 3^4.
\end{aligned}$$

again in agreement with Theorem 4.

## 11. CHROMATIC EVALUATION OF SPIN NETWORKS

The purpose of this section is to prove the rationality statement of the spin generating function stated in Theorem 5 of Section 4. We will use the *chromatic evaluation method* which goes back to [Pe2]. Our proof builds on earlier work by [We] and [KL] on planar spin networks.

The chromatic method for the evaluation of a spin network is based on introducing a generalized evaluation  $\langle \Gamma, \gamma \rangle_N^P$  following the algorithm of Definition 1.1, with the exception that a collection of  $n$  loops is assigned the value  $N^n$  instead of  $(-2)^n$ . Therefore,

$$\langle \Gamma, \gamma \rangle_{-2}^P = \langle \Gamma, \gamma \rangle^P.$$

Now the crucial observation is that  $\langle \Gamma, \gamma \rangle_N^P$  is a polynomial in  $N$  with integer coefficients, so that  $\langle \Gamma, \gamma \rangle^P$  is determined once we know the values of  $\langle \Gamma, \gamma \rangle_N^P$  for all positive  $N$ .

For positive  $N$  there exists an alternative interpretation of  $\langle \Gamma, \gamma \rangle_N^P$  as a contraction of tensors. For convenience we will orient the edges of  $\Gamma$ . Let  $V$  be an  $N$  dimensional vector space and for  $\sigma \in S_n$  define the following permutation operators  $\sigma : V^{\otimes n} \rightarrow V^{\otimes n}$ ,  $\sigma(e_{i_1} \otimes \dots \otimes e_{i_n}) = e_{\sigma(i_1)} \otimes \dots \otimes e_{\sigma(i_n)}$ . At every edge colored  $n$  we can now speak of the map  $\sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \sigma$ . Note that the single arcs in the expanded diagram for  $(\Gamma, \gamma)$  now correspond to copies of  $V$ . The systems of arcs at the vertices indicate how to contract (compose) the tensors on the edges. Since all the tensors involved are linear combinations of Kronecker deltas we can for each choice of permutations count the number of contracted deltas and this will exactly contribute a term  $\pm N^k$  for some  $k$  since the trace of a delta is simply  $N$ . This coincides with the counting

and evaluation of loops in every state, since a contracted delta represents a loop. However there is also another way to carry out this tensor calculation.

We can also calculate  $\langle \Gamma, \gamma \rangle_N^P$  by writing out the contraction of tensors into the single contributions of basis elements in the various tensor products. This will yield an explicit formula for the evaluation of a general spin network in terms of so called curve configurations on  $\Gamma$ . Recall that a curve is a 2-regular subgraph of  $\Gamma$  and a curve configuration is a function from the set  $C$  of all curves to the natural numbers,  $\mathcal{L} : C \rightarrow \mathbb{N}$ . For such functions we define  $|\mathcal{L}|$  to be the sum of the function values. Define sign  $(-1)^{\iota(\mathcal{L})}$ , where for any subset  $\iota(\mathcal{L})$  is the minimal number of intersections one can have when drawing  $\mathcal{L}(c)$  copies of  $c$  for every curve on the thickened ribbon graph  $\Gamma$ .

**Lemma 11.1.** *For every spin network  $(\Gamma, \gamma)$  we have*

$$\langle \Gamma, \gamma \rangle = \sum_{\gamma_{\mathcal{L}} = \gamma} (-1)^{\iota(\mathcal{L}) + |\mathcal{L}|} (|\mathcal{L}| + 1) \binom{|\mathcal{L}|}{\mathcal{L}} \in \mathbb{Z}.$$

This implies in particular that the generating series  $S_{\Gamma}$  of Definition 4.1 satisfies  $S_{\Gamma} \in \mathbb{Z}[[z_e, e \in E(\Gamma)]]$  proving one part of Theorem 5.

*Proof.* (of Lemma 11.1) We will calculate  $\langle \Gamma, \gamma \rangle_N^P$  by writing out the contraction of tensors into the single contributions of basis elements in the various tensor products. Given a choice of permutations we place tensor products of basis elements  $e_{i_1} \otimes \cdots \otimes e_{i_n}$  on every edge. Such a term contributes 0 if some basis vectors on the same strand are unequal or the same basis element occurs twice in a tensor product corresponding to an edge. Otherwise the contribution is always  $(-1)^{\#\text{crossings}}$ .

To enumerate the non-zero states in an orderly fashion including the signs of their contributions we proceed in three stages. First we choose which sets of arcs get labeled by the same base vector. Then we assign base vectors to the loops in the curves. Finally we count the number of such labeled states.

In the first stage we choose the collections of all loops that are labeled by the same base vector. For non-zero states these collections are necessarily 2-regular subgraphs of  $\Gamma$ , also known as curves. We will view curves both as sets of the loops they contain and as subgraphs of  $\Gamma$ . This first stage of the calculation is summarized in the concept of a curve configuration. To every curve  $c \in C_{\Gamma}$  we assign the number of distinct base vectors it occurs with. The curve configuration  $\mathcal{L}$  determines the total coloring of the spin network  $\gamma$ . However there may be many curve configurations yielding the same coloring  $\gamma$ .

Note that all states with the same curve configuration contribute the same sign. This is because of planarity of the (multiple) annulus neighborhood of the curve: two loops in the same curve can only cross an even number of times. The common sign is determined by the cyclic ordering of the graph and equals  $(-1)^{\iota(\mathcal{L})}$ . This follows directly from a count of the number of intersections of such states.

In the second stage we choose base vectors and we assign them to the loops such that all the loops with the same base vector constitute exactly one copy of a curve. First we need to choose which base vectors to use. There are  $\binom{N}{|\mathcal{L}|}$  ways to do this. Then we need distribute these over the various curves, assigning  $\mathcal{L}(c)$  base vectors to the curve  $c$  in no particular order. There are  $\binom{|\mathcal{L}|}{\mathcal{L}}$  ways to do this.

In the last stage we need to count how many arrangements of loops are possible given a curve configuration with base vectors at every loop. As was shown in [KL] there are  $\mathcal{I}!$  ways to do this. Their argument does not make use of planarity but for completeness sake we still reproduce the argument here. If we ignore the labelling by base vectors we will count exactly  $\frac{1}{\mathcal{I}!}$  of the possible states. However if we also choose to ignore the fact that in contributing states the loops cannot self intersect, then counting possibly zero states will multiply our count by the same factor  $\mathcal{L}!$ . To see why, note that to modify a zero state takes one permutation per curve.

Recall that  $\mathcal{I}!$  corresponds to a choice of a permutation on the arcs on each side of every vertex in the ribbon graph. We will show that there are exactly  $\mathcal{I}!$  (possibly zero) states corresponding to a given curve configuration. First choose one such a state  $S$ . We will show that all others can be reached by inserting a unique permutation at every side of every vertex, with the appropriate modification. At any rate there cannot be more than  $\mathcal{I}!$  such states, because arcs belonging to one side of a vertex can never be permuted to become

part of another side, because that would destroy the curve configuration. Even inserting a permutation at the side of a vertex may destroy the curve configuration, but in this case this can be compensated as follows. Choose an edge adjacent to the chosen side of the vertex and insert the inverse of the part of the permutation swapping arcs belonging to different curves. Exactly because of the fact that this compensating permutation involves only arcs corresponding to unequal curves this does not interfere with any other inserted permutations at the sides of vertices or their compensations.

Summarizing, we get

$$\langle \Gamma, \gamma \rangle_N^P = \sum_{\gamma_{\mathcal{L}} = \gamma} (-1)^{\iota(\mathcal{L})} \binom{N}{|\mathcal{L}|} \binom{|\mathcal{L}|}{\mathcal{L}} \mathcal{I}!$$

and rescaling we get

$$\langle \Gamma, \gamma \rangle_N = \sum_{\gamma_{\mathcal{L}} = \gamma} (-1)^{\iota(\mathcal{L})} \binom{N}{|\mathcal{L}|} \binom{|\mathcal{L}|}{\mathcal{L}}$$

Finally setting  $N = -2$  provides a general formula for spin network evaluation as given in the lemma.  $\square$

To find a generating function for these evaluations, we first introduce a more general generating function.

**Definition 11.2.** Define the *curve generating function*  $CG_{\Gamma}$  of a graph  $\Gamma$  by

$$CG_{\Gamma} = \sum_{\mathcal{L}} \langle \mathcal{L} \rangle \prod_{c \in \mathcal{C}} w_c^{\mathcal{L}(c)}$$

where  $CG_{\Gamma}$  is a function of  $|C_{\Gamma}|$  variables, one for each curve in  $\Gamma$ . The exponents of these variables encode a curve configuration, and the evaluation  $\langle \mathcal{L} \rangle$  of a curve configuration  $\mathcal{L}$  is given by

$$\langle \mathcal{L} \rangle = (-1)^{\iota(\mathcal{L}) + |\mathcal{L}|} (|\mathcal{L}| + 1) \binom{|\mathcal{L}|}{\mathcal{L}}$$

The spin generating function  $S_{\Gamma}$  can be recovered from the curve generating function  $CG_{\Gamma}$  by substituting  $w_c = \prod_{e \in c} z_e$ , because the color of an edge is the sum of the number of curves (with multiplicity) containing that edge.

*Proof.* (Of Theorem 5) Our goal now is to write  $CG_{\Gamma}$  and hence  $S_{\Gamma}$  as a rational function. As a first step we note that if we drop the term  $(-1)^{\iota(\mathcal{L})}$  then we can sum the resulting series using the binomial series and the multinomial theorem to get

$$\sum_{\mathcal{L}} (-1)^{|\mathcal{L}|} (|\mathcal{L}| + 1) \binom{|\mathcal{L}|}{\mathcal{L}} w^{\mathcal{L}} = \frac{1}{(1 + w_1 + \dots + w_{C_{\Gamma}})^2} = Q(w_1, \dots, w_{C_{\Gamma}}) = \sum_{\mathcal{L}} b_{\mathcal{L}} w^{\mathcal{L}}$$

Using this together with the Fourier expansion of  $(-1)^{\iota(\mathcal{L})}$  viewed as a function on the group  $(C_2)^{C_{\Gamma}}$  as

$$(-1)^{\iota(\mathcal{L})} = \sum_{X \subset C_{\Gamma}} a_X (-1)^X$$

we find

$$CG_{\Gamma} = \sum (-1)^{\iota(\mathcal{L})} b_{\mathcal{L}} w^{\mathcal{L}} = \sum_{X \subset C_{\Gamma}} a_X \sum_{\mathcal{L}} (-1)^{X \cap \mathcal{L}} b_{\mathcal{L}} w^{\mathcal{L}} = \sum_{X \subset C_{\Gamma}} a_X Q_X$$

where  $Q_X(w_1, \dots, w_{C_{\Gamma}}) = Q(v_1, \dots, v_{C_{\Gamma}})$  and  $v_i = \epsilon_X(c_i) w_i$ .  $\square$

## 12. CHALLENGES AND FUTURE DIRECTIONS

In this section we list some challenges and future directions. Recall from Definition 2.3 that every spin network  $(\Gamma, \gamma)$  (and consequently every ribbon graph) has a spectral radius  $\rho_{\Gamma, \gamma}$  and a number field  $K_{\Gamma, \gamma}$ . Our first problem implies the existence of an upper bound of the spectral radius in terms of  $\Gamma$  alone. This is phrased more conveniently in terms of the unitary evaluation of Definition 9.4.

**Problem 2.** Show that the unitary evaluation of a spin network  $(\Gamma, \gamma)$  satisfies

$$|\langle \Gamma, \gamma \rangle^U| \leq 1$$

This problem may be proven using unitarity and locality in a way similar to the proof that the Reshetikhin-Turaev invariants of a closed 3-manifold grow at most polynomially with respect to the level; see [Ga5, Thm.2.2].

Recall the spectral radius of a cubic ribbon graph from Definition 2.3; it is a nonnegative algebraic number. Our next problem is a version of the Volume Conjecture for classical spin networks with all edges colored by 2.

**Problem 3.** Show that the spectral radius of a cubic ribbon graph with  $2n$  vertices equals to  $3^{3n}$ .

Theorem 4 confirms this problem for the following ribbon graphs: the  $\Theta$ , the tetrahedron, the 3-faced drum and more generally for the infinite family of ribbon graphs of Remark 3.6. There is convincing numerical evidence for the above problem in the case of the  $K_{3,3}$  and the Cube.

Recall the generating series  $F_{\Gamma,\gamma}(z)$  of Equation (4). Problem 3 motivates the next problem.

**Problem 4.** Give a geometric meaning to the finitely many singularities of the generating series  $F_{\Gamma,\gamma}(z)$  for a spin network  $(\Gamma, \gamma)$ .

At the moment, we do not know the answer to Problem 4 for the Cube with all edges colored by 2. Numerical evidence suggests that there are 12 equally spaced singularities in the boundary of the disk of convergence of  $F_{\text{Cube},2}(z)$ .

The next problem is a computational challenge to all the known asymptotic methods, and shows their practical limitations.

**Problem 5.** Compute the asymptotics of the evaluation  $\langle K_{3,3}, 2n \rangle$  (given explicitly in Proposition 6.5) and  $\langle \text{Cube}, 2n \rangle$ .

**Problem 6.** Give a geometric interpretation of the number field  $K_{\Gamma,\gamma}$  of a spin network  $(\Gamma, \gamma)$ .

In Theorem 4 we identified the Stokes constants of the  $6j$ -symbol with geometric quantities of the associated metric tetrahedron.

**Problem 7.** Give a geometric meaning to the Stokes constants of the sequence  $\langle \Gamma, n\gamma \rangle$ .

The next problem is formulated by several examples involving the WZ method, discussed in Section 10. A positive answer can give an independent proof of Theorem 4.

**Problem 8.** Prove that for every coloring  $\gamma$  of the tetrahedron spin network  $(\triangle, \gamma)$ , the sequence  $\langle \triangle, n\gamma \rangle$  satisfies a second order recursion relation with coefficients polynomials in  $n$ . Can you compute the coefficients of this recursion from  $\gamma$  alone?

Our last problem concerns an integrality property of quantum spin networks. Our paper discusses four normalizations  $\langle \Gamma, \gamma \rangle^P$ ,  $\langle \Gamma, \gamma \rangle$ ,  $\langle \Gamma, \gamma \rangle^B$  and  $\langle \Gamma, \gamma \rangle^U$  for the evaluation of a classical spin network  $(\Gamma, \gamma)$ ; see Definitions 1.1, 1.2, 6.2 and 9.4. With the notation of these definitions, these normalizations are related by

$$(57) \quad \langle \Gamma, \gamma \rangle^P = \mathcal{I}! \langle \Gamma, \gamma \rangle, \quad \langle \Gamma, \gamma \rangle^B = \frac{\mathcal{I}!}{\mathcal{E}!} \langle \Gamma, \gamma \rangle, \quad \langle \Gamma, \gamma \rangle^U = \frac{1}{\Theta(\gamma)} \langle \Gamma, \gamma \rangle.$$

On the other hand, if  $(\Gamma, \gamma)$  is a quantum spin network, the Kauffman bracket evaluation  $\langle \Gamma, \gamma \rangle^B$  makes sense, and in general it is a rational function of the bracket parameter  $A$ ; see Definition 6.2 and [KL]. The following problem is a first step towards the categorification of the evaluation of a quantum spin network.

**Problem 9.** Show that for every quantum spin network  $(\Gamma, \gamma)$ , the evaluation  $\langle \Gamma, \gamma \rangle^B \mathcal{E}! / \mathcal{I}!$  lies in  $\mathbb{Z}[A^{\pm 1}]$ .

One can show that the above problem holds for the  $\Theta$  and the tetrahedron spin networks, or more generally for the class of quantum spin networks of Remark 3.6.

Let us end this section with a remark. The main results of our paper can be extended to evaluations of spin networks corresponding to higher rank Lie groups. This will be discussed in a later publication.

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