Math. 1501N5, Test 2, 11/2/1999

Solutions:

1. (4 pts) Calculate

\[ \frac{d}{dx} \left[ \cos \left( \frac{3x + 1}{1 + 2x^3} \right)^5 + \tan x \right] \]

\[ = -\sin \left( \frac{3x + 1}{1 + 2x^3} \right)^5 + \tan x \right] \times \left[ 5 \left( \frac{3x + 1}{1 + 2x^3} \right)^4 \frac{3(1 + 2x^3) - 6x^2(3x + 1) + \sec^2 x}{(1 + 2x^3)^2} \right]. \]

2. (5 pts) Find the equation of the line tangent to the curve \( \sin y + x^3y^4 = 0 \) at (1, 0).

Differentiate the equation implicitly to get

\[ \cos y \frac{dy}{dx} + 3x^2y^4 + x^3y^3 \frac{dy}{dx} = 0. \]

If \( x = 1 \) and \( y = 0 \) we then have \( \frac{dy}{dx} = 0 \) and so the equation of the tangent line is \( y = 0 \).

3. (5 pts) A poster is to have an area of 180 in\(^2\) with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

Let \( x \) be the length of the side of the poster. Then the length of the top (and the bottom) is \( 180/x \). Therefore the printed area as a function of \( x \) is

\[ A(x) = 180 - 2x - \left( \frac{180}{x} - 2 \right) - 2 \left( \frac{180}{x} - 2 \right) = 186 - 2x - \frac{540}{x}, \]

where \( 3 \leq x \leq 90 \). We need to find \( x \) maximizing \( A \).

\[ A'(x) = -2 + \frac{540}{x^2} = 0 \]

if and only if \( x = \sqrt{270} \). Since \( A(3) = A(90) = 0 \) the function must have maximum at \( x = \sqrt{270} \).

4. (5 pts) How many solutions does the equation \( 4x^3 - 4x^2 + x = 0 \) have in the interval \([-1, 1]\)?

Let \( f(x) = 4x^3 - 4x^2 + x \). \( f'(x) = 12x^2 - 8x + 1 \) and so \( f'(x) = 0 \) if \( x = 1/2, x = 1/6 \). Now: \( f(-1) < 0, f(1/6) > 0, f(1/2) = 0, f(1) > 0 \). Therefore by the Intermediate Value Theorem \( f \) has one zero in the interval \((-1, 1/6)\). The second zero is at \( 1/2 \). There are no other zeros since if there were one there would have to be another critical number. Therefore the equation has two solutions.

5. (5 pts) A lighthouse is located on a small island 3 km away from the nearest point \( P \) on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from \( P \)?
Let $x$ be the distance along the shoreline between the light and the point $P$, and $\theta$ be the angle between the line connecting the lighthouse with $P$ and the light beam. We know that $d\theta/dt = 8\pi$ (rad/min). Since $x = 3 \tan \theta$ we get

$$\frac{dx}{dt} = 3 \sec^2 \theta \frac{d\theta}{dt}.$$ 

When $x = 1 \tan \theta = 1/3$ and thus $\sec^2 \theta = 10/9$. Therefore when $x = 1$

$$\frac{dx}{dt} = 3 \frac{10}{9} 8\pi = \frac{80\pi}{3} \text{ (km/min)}.$$ 

6. (16 pts) Let

$$f(x) = \frac{1}{x^3 - x}.$$ 

(a) Find the critical numbers of $f$ and the intervals on which $f$ is increasing or decreasing.
(b) Find the local maximum and minimum values of $f$.
(c) Find the intervals of concavity and inflection points.
(d) Find the asymptotes of $f$ (if any).
(e) Use the information from parts (a)-(d) to sketch the graph of $f$.

(a) Domain of $f$ is $\mathbb{R} \setminus \{0, \pm 1\}$. Moreover $f$ is differentiable on its domain and is odd and so its graph is symmetric about the origin.

$$f'(x) = \frac{1 - 3x^2}{(x^3 - x)^2}.$$ 

Therefore $f(x)' = 0$ if $x = \pm 1/\sqrt{3}$ and these are the critical numbers.

$f' < 0$ (and so $f$ is decreasing) on

$$(-\infty, -1) \cup (-1, -\frac{1}{\sqrt{3}}) \cup \left(\frac{1}{\sqrt{3}}, 1\right) \cup (1, +\infty),$$

$f' > 0$ (and so $f$ is increasing) on

$$\left(-\frac{1}{\sqrt{3}}, 0\right) \cup (0, \frac{1}{\sqrt{3}}).$$

(b) Since $f'$ changes sign at $\pm 1/\sqrt{3}$ we see that there is a local minimum at $-1/\sqrt{3}$ and a local maximum at $1/\sqrt{3}$.

(c) 

$$f''(x) = \frac{6x^4 - 3x^2 + 1}{x^3(x^2 - 1)^3}.$$
Since \(6x^4 - 3x^2 + 1\) is always greater than 0, \(f''\) is never zero and so there are no inflection points. We have:

\(f'' < 0\) (and thus \(f\) is concave down) on \((-\infty, -1) \cup (0, 1)\).

\(f'' > 0\) (and thus \(f\) is concave up) on \((-1, 0) \cup (1, +\infty)\).

(d) \[\lim_{x \to \pm\infty} f(x) = 0, \quad \lim_{x \to -1^-} f(x) = -\infty, \quad \lim_{x \to -1^+} f(x) = +\infty, \quad \lim_{x \to 0^-} f(x) = +\infty, \quad \lim_{x \to 0^+} f(x) = -\infty, \quad \lim_{x \to 1^-} f(x) = -\infty, \quad \lim_{x \to 1^+} f(x) = +\infty.\]

Therefore \(f\) has vertical asymptotes \(x = -1, x = 0, \) and \(x = 1,\) and a horizontal asymptote \(y = 0.\)

(e) Please sketch the graph yourself.