1. Find the area of the parallelogram in $\mathbb{R}^2$ with vertices $A = (0, 0), B = (2, 1), C = (3, 4), D = (1, 3)$. What is the angle between the diagonal $AC$ and the line passing through the points $A$ and $E = (1, -1)$?

The area of the parallelogram is equal to the absolute value of the determinant of the matrix whose columns are vectors generating the parallelogram. So

$$\text{Area} = \left| \det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \right| = 5.$$

As for the angle the vector

$$\vec{AC} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \vec{AE} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and so if $\alpha$ is the angle we have

$$\cos \alpha = \frac{\vec{AC} \cdot \vec{AE}}{||\vec{AC}|| ||\vec{AE}||} = \frac{3 - 4}{5 \cdot \sqrt{2}} = \frac{-1}{5\sqrt{2}}.$$

Therefore $\alpha = \arccos \left( \frac{-1}{5\sqrt{2}} \right)$.

2. Solve the system

\[
\begin{align*}
x + y + 2z &= 9 \\
2x + 4y - 3z &= 1 \\
3x + 6y - 5z &= 0
\end{align*}
\]

The augmented matrix for the system is

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix}$$

We add $-2$ times the first row to the second to obtain

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{pmatrix}.$$

We add $-3$ times the first row to the third to obtain

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{pmatrix}.$$
We multiply the second row by $1/2$ to obtain

\[
\begin{pmatrix}
1 & 1 & 2 & 9 \\
0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\
0 & 3 & -11 & -27
\end{pmatrix}.
\]

We add $-3$ times the second row to the third to obtain

\[
\begin{pmatrix}
1 & 1 & 2 & 9 \\
0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\
0 & 0 & -\frac{1}{2} & -\frac{3}{2}
\end{pmatrix}.
\]

From this we get $z = 3$ and then (looking at the second row) $y = 2$ which finally gives $x = 1$.

3. Are vectors

\[
\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}
\]

linearly independent?

Three vectors in $\mathbb{R}^3$ are linearly independent if and only if the determinant of a matrix whose columns are the vectors is nonzero. Here we have

\[
\det \begin{pmatrix}
2 & 1 & -1 \\
3 & 2 & 1 \\
1 & 3 & 1
\end{pmatrix} = 2(2 - 3) - 3(1 + 3) + 1(1 + 2) = -11.
\]

So the vectors are linearly independent.

4. Give an example of a $5 \times 6$ matrix whose null space has dimension 2.

For instance

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

This matrix has 4 pivotal columns so the dimension of the column space is 4. By the law of conservation of dimension, the dimension of the null space is $6 - 4 = 2$.

5. The matrix

\[
A = \begin{pmatrix}
1 & -2 & 0 & 0 & 3 \\
2 & -5 & -3 & -2 & 6 \\
0 & 5 & 15 & 10 & 0 \\
2 & 6 & 18 & 8 & 6
\end{pmatrix}
\]
row reduces to

\[ \tilde{A} = \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

(a) Find a basis for the column space of \( A \).

\( \tilde{A} \) has pivotal ones in columns 1, 2 and 3 and so columns 1, 2 and 3 of \( A \) create a basis for the column space of \( A \).

(b) Find a basis for the null space of \( A \).

The null space has dimension 2. Columns 4 and 5 are nonpivotal in \( \tilde{A} \). Therefore we are looking for vectors of the form

\[ \vec{v}_1 = \begin{bmatrix} \cdot \\ \cdot \\ 1 \\ 0 \\ \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}, \]

where dots indicate unknown numbers. Multiplying \( \tilde{A} \) by the first one we get

\[
\begin{align*}
x_1 - 2x_2 &= 0 \\
x_2 + 3x_3 + 2 &= 0 \\
x_3 + 1 &= 0.
\end{align*}
\]

This gives \( x_3 = -1, x_2 = 1, x_1 = 2 \) and so

\[ \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}. \]

Now multiplying \( \tilde{A} \) by \( \vec{v}_2 \) we get

\[
\begin{align*}
x_1 - 2x_2 + 3 &= 0 \\
x_2 + 3x_3 &= 0 \\
x_3 &= 0.
\end{align*}
\]
This gives \( x_3 = 0, x_2 = 0, x_1 = -3 \) and so
\[
\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

6. Find the least squares solution of \( A \vec{x} = \vec{b} \) if
\[
A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{pmatrix}, \quad \vec{b} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}.
\]
The matrix
\[
A^T A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}
\]
is invertible. Therefore the least squares solution is given by the formula
\[
\vec{x} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{14} \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{1}{2} \end{bmatrix}.
\]

7. Find the matrix \([T]\) corresponding to the linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by
\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_2 + x_3 \\ -5x_1 + 13x_2 \\ x_1 - 2x_3 \end{bmatrix}.
\]
Either by inspection or by looking at images of the vectors of the standard basis of \( \mathbb{R}^3 \) we get
\[
[T] = \begin{pmatrix} 0 & 1 & 1 \\ -5 & 13 & 0 \\ 1 & 0 & -2 \end{pmatrix}.
\]