1. (4 pts) Let \( u(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \) and \( v(t) = \sqrt{t}\mathbf{i} + t\mathbf{j} + \mathbf{k}. \) Find

\[
\int_0^1 (u(t) \times v(t))dt
\]

\[
u(t) \times v(t) = (t - t^3)\mathbf{i} - (1 - t^2)\mathbf{j} + (t - t^2)\mathbf{k}
\]

Therefore

\[
\int_0^1 (u(t) \times v(t))dt = \left[ \left( \frac{1}{2}t^2 - \frac{1}{4}t^4 \right)\mathbf{i} + \left( \frac{2}{7}t^7 - t \right)\mathbf{j} + \left( \frac{1}{2}t^2 - \frac{2}{5}t^5 \right)\mathbf{k} \right]_0^1 = \frac{1}{4}\mathbf{i} - \frac{5}{7}\mathbf{j} + \frac{1}{10}\mathbf{k}.
\]

2. (6 pts) Find the osculating plane of the curve \( r(t) = \sin^2 t\mathbf{i} + t\mathbf{j} + \cos^2 t\mathbf{k} \) at the point \((0, 0, 1)\)

We have \( r'(t) = 2\cos 2t\mathbf{i} + \mathbf{j} - 2\sin 2t\mathbf{k}, \) \( ||r'(t)|| = \sqrt{5}, \) and then

\[
T(t) = \frac{1}{\sqrt{5}}(2\cos 2t\mathbf{i} + \mathbf{j} - 2\sin 2t\mathbf{k})
\]

and

\[
T'(t) = \frac{1}{\sqrt{5}}(-4\sin 2t\mathbf{i} - 4\cos 2t\mathbf{k}).
\]

The vector \( r'(t) \) is tangent to the curve (even though it does not have unit length) and the vector \( T'(t) \) is in the direction of the principal normal (even though it is not the principal normal.) Since \( r(0) = (0, 0, 1) \) the osculating plane is spanned by the vectors \( r'(0) = (2, 1, 0), \) and \( \sqrt{5}T'(0) = (0, 0, -4). \) Taking their cross product we obtain that \((-4, -8, 0)\) and hence \((-1, -2, 0)\) is perpendicular to the osculating plane. Its equation is thus \(-x - 2y + 0(z - 1) = 0, i.e.

\[
x + 2y = 0.
\]

3. (5 pts) Find the length of the curve parametrized by

\[
r(t) = \frac{4}{3}t^\frac{3}{2}\mathbf{i} + \sqrt{2}t\mathbf{j} + t^\frac{1}{2}\mathbf{k}, \quad 1 \leq t \leq 2.
\]

\[
r'(t) = 2\sqrt{t}\mathbf{i} + \sqrt{2}\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}
\]

so

\[
L = \int_1^2 \sqrt{4t + 2 + \frac{1}{4t}} \, dt = \int_1^2 \sqrt{(2\sqrt{t} + \frac{1}{2\sqrt{t}})^2} \, dt
\]
\[
\int_1^2 (2\sqrt{t} + \frac{1}{2\sqrt{t}}) dt = \left(\frac{4}{3} t^{\frac{3}{2}} + \frac{1}{2} t^{\frac{1}{2}}\right)|_1^2 = \left(\frac{4}{3} \cdot 2^{\frac{3}{2}} + 2^{\frac{1}{2}}\right) - \left(\frac{4}{3} + 1\right) = \frac{11}{3}\sqrt{2} - \frac{7}{3}.
\]

4. (6 pts) The position of an object is given by \( \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t^2 \mathbf{k} \). Its acceleration \( \mathbf{a} \) can be decomposed into \( \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \). Find \( a_T \) and \( a_N \).

We compute 
\[
\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + 2t \mathbf{k}, \quad \frac{ds}{dt} = ||\mathbf{r}'(t)|| = \sqrt{1 + 4t^2}.
\]
Therefore 
\[
a_T = \frac{d^2s}{dt^2} = \frac{d}{dt} \sqrt{1 + 4t^2} = \frac{4t}{\sqrt{1 + 4t^2}}.
\]
Now 
\[
\mathbf{a}(t) = \mathbf{r}''(t) = -\sin t \mathbf{i} - \cos t \mathbf{j} + 2 \mathbf{k}, \quad ||\mathbf{a}(t)||^2 = 5.
\]
Using the fact that \( ||\mathbf{T}||, ||\mathbf{N}|| = 1 \), and that \( a_N > 0 \) we therefore have 
\[
a_N = \sqrt{||\mathbf{a}||^2 - ||a_T||^2} = \sqrt{5 - \frac{16t^2}{1 + 4t^2}} = \sqrt{\frac{5 + 4t^2}{1 + 4t^2}}.
\]

5. (5 pts) Find the curvature of the curve \( y = x^7 + x - 1 \) at the point \( (1, 1) \).

The curve is a graph. We compute 
\[
y' = 7x^6 + 1, \quad y'' = 42x^5.
\]
Using the formula for the curvature of the graph we therefore obtain 
\[
k = \frac{|y''|}{(1 + |y'|^2)^{\frac{3}{2}}} = \frac{42x^5}{(1 + (7x^6 + 1)^2)^{\frac{3}{2}}}.
\]
When \( x = 1 \) the curvature is 
\[
k = \frac{42}{65^{\frac{3}{2}}}.
\]

6. (4 pts) Find the domain and the range of the function \( f(x, y, z) = \ln(2 - \ln x) + \sqrt{y^3 z} \) and find the equation of the level surface of \( f \) that contains the point \( (e, 1, 1) \).

We must have \( 2 - \ln x > 0 \) and \( y^3 z \geq 0 \). This gives \( 0 < x < e^2 \) and either \( y, z \geq 0 \) or \( y, z \leq 0 \), i.e. 
\[
D = \{(x, y, z) : 0 < x < e^2, \text{ and either } y, z \geq 0 \text{ or } y, z \leq 0\}.
\]
The range is equal to the whole real line since for any real number \( c \)
\[
f(e^{2-e^c}, 0, 0) = \ln(2 - (2 - e^c)) + 0 = c.
\]
Since \( f(e, 1, 1) = \ln(2 - \ln e) + \sqrt{1} = \ln 1 + 1 = 1 \), the level surface of \( f \) that contains the point \( (e, 1, 1) \) has the equation 
\[
\ln(2 - \ln x) + \sqrt{y^3 z} = 1.
\]