1. (5 pts) Find a function \( f \) such that
\[
\nabla f(x, y) = (3x^2 y^2 + 1)i + (2x^3 y + 3y^2 - \sin y)j
\]
if it exists.

\[
\frac{\partial (3x^2 y^2 + 1)}{\partial y} = 6x^2 y = \frac{\partial (2x^3 y + 3y^2 - \sin y)}{\partial x}
\]

so the vector field is a gradient vector field. Now

\[
f(x, y) = \int (3x^2 y^2 + 1) \, dx = x^3 y^2 + x + g(y)
\]

and then

\[
2x^3 y + 3y^2 - \sin y = \frac{\partial (x^3 y^2 + x + g(y))}{\partial y} = 2x^3 y + g'(y).
\]

This implies \( g'(y) = 3y^2 - \sin y \) so \( g(y) = y^3 + \cos y + c \). Therefore the functions \( f \) have the form

\[
f(x, y) = x^3 y^2 + x + y^3 + \cos y + c.
\]

2. (5 pts) Find the \( y \)-coordinate of the center of mass of a lamina that occupies the region bounded by the \( x \)-axis and the parabola \( y = 1 - x^2 \) if the density function is \( \lambda(x, y) = x^2 \).

\[
\text{Mass} = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} x^2 \, dy \, dx = \int_{-1}^{1} (x^2 y) \bigg|_{y=0}^{y=1-x^2} \, dx
\]

\[
= \int_{-1}^{1} (x^2 - x^4) \, dx = \left(\frac{x^3}{3} - \frac{x^5}{5}\right) \bigg|_{-1}^{1} = \frac{4}{15}
\]

The \( y \)-coordinate of the center of mass is equal to

\[
y_M = \frac{15}{4} \left(\int_{-1}^{1} x^2 \, dy \right) \int_{0}^{\sqrt{1-x^2}} x^2 \, dy
\]

\[
= \frac{15}{8} \left(\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7}\right) \bigg|_{-1}^{1} = \frac{15}{4} \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{2}{7}.
\]

3. (5 pts) Find the volume of the region cut from the solid cylinder \( x^2 + y^2 \leq 1 \) by the sphere \( x^2 + y^2 + z^2 = 4 \).

Denote the region by \( V \). The volume of \( V \) is equal to

\[
\int \int \int_V dxdydz.
\]
Integrating in cylindrical coordinates we obtain
\[
\int \int \int_V dxdydz = \int_0^{2\pi} \int_{\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dz d\theta dr = 2\pi \int_0^1 2r\sqrt{4-r^2} dr
\]
\[=-2\pi \frac{2}{3} (4-r^2)^{\frac{3}{2}} \bigg|_0^1 = \frac{4\pi}{3} (8-3\sqrt{3}).\]

4. (5 pts) Express
\[
\int \int \int_V f(x, y, z) dxdydz,
\]
where \(V\) is the region in the first octant bounded by the surfaces \(z = 0, y = x^2, \) and \(y + z = 1,\) as the iterated integral
\[
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-z} f(x, y, z) dy dz dx.
\]
The iterated integral is
\[
\int_1^0 \int_{-1}^{x} \int_{x^2}^{1-z} f(x, y, z) dy dz dx.
\]

5. (5 pts) Evaluate
\[
\int_{-3}^0 \int_{\sqrt{9-x^2}}^0 \int_{\sqrt{9-x^2-y^2}}^0 z\sqrt{x^2+y^2+z^2} dz dy dx.
\]
The region of integration is one eight of the ball centered at the origin with radius 3. The \(x, y\)-coordinates are in the second quadrant of the \(xy\)-plane and the \(z\)-coordinate is negative. Therefore in spherical coordinates the integral is equal to
\[
\int_0^3 \int_{\frac{\pi}{2}}^\pi \int_{\frac{\pi}{2}}^\pi \rho^2 \cos \phi \rho^2 \sin \phi d\phi d\theta d\rho
\]
\[= \int_0^3 \rho^4 d\rho \int_{\frac{\pi}{2}}^\pi d\theta \int_{\frac{\pi}{2}}^\pi \frac{\sin 2\phi}{2} d\phi
\]
\[= \frac{\rho^5}{5} \bigg|_0^3 \left( \frac{\pi}{2} \right) \left( -\frac{\cos 2\phi}{4} \right) \bigg|_\frac{\pi}{2}^\pi = \frac{243\pi}{10} \left( -\frac{1}{2} \right) = -\frac{243\pi}{20}.
\]

6. (5 pts) Use the change of variables \(u = y - x, v = y + x\) to evaluate
\[
\int \int_\Omega \cos \left( \frac{y-x}{y+x} \right) dxdy,
\]
where \( \Omega \) is the trapezoidal region with vertices \((1,0),(2,0),(0,1)\) and \((0,2)\).

In the new variables \( u = y - x, v = y + x \) we have

\[
x = \frac{v - u}{2}, \quad y = \frac{u + v}{2}.
\]

The Jacobian of this transformation is

\[
J(u,v) = \det \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{1}{2}.
\]

We need to find the image of \( \Omega \) under the map \( u = y - x, v = y + x \). This is a linear map from the \( xy \)-plane to the \( uv \)-plane so to find the image of \( \Omega \) under this map we only need to find the images of the vertices of \( \Omega \). The vertices \((1,0),(2,0),(0,1)\) and \((0,2)\) are being mapped respectively onto \((-1,1),(-2,2),(1,1)\) and \((2,2)\). So the image is the trapezoidal region \( S = \{ (u,v) : 1 \leq v \leq 2, -v \leq u \leq v \} \). Therefore

\[
\int \int_{\Omega} \cos \left( \frac{y - x}{y + x} \right) \, dx \, dy = \int_{1}^{2} \int_{-v}^{v} \cos \left( \frac{u}{v} \right) |J(u,v)| \, du \, dv = \frac{1}{2} \int_{1}^{2} v \sin \left( \frac{u}{v} \right) \bigg|_{u=-v}^{u=v} \, dv \\
= \frac{1}{2} \int_{1}^{2} v (\sin 1 - \sin(-1)) \, dv = \frac{\sin 1}{2} v^2 \bigg|_{1}^{2} = \frac{3}{2} \sin 1.
\]