Math. 2403, Test1, 2/7/2008  
Name: SOLUTIONS

Please work carefully, show your work and justify your answers. Each problem is worth 6 points.

1. (a) Draw the slope field of
\[ \frac{dy}{dx} = y(1 - y)(2 - y). \]

(b) Draw the phase portrait and determine the stability of equilibria.

Equilibria are at 0, 1, and 2. 1 is stable and 0, 2 are unstable.

2. Find the general solution of the differential equation
\[ \frac{dy}{dt} + \frac{t}{t^2 + 1} y = \frac{t}{t^2 + 1}. \]

This is a linear equation. The integrating factor is
\[ \mu(t) = e^{\int \frac{t}{t^2 + 1} dt} = e^{\frac{1}{2} \ln(t^2 + 1)} = \sqrt{t^2 + 1}. \]

Therefore we get
\[ \sqrt{t^2 + 1} y = \int \sqrt{t^2 + 1} \frac{t}{t^2 + 1} dt = \int \frac{t}{\sqrt{t^2 + 1}} dt = \sqrt{t^2 + 1} + c \]
so finally
\[ y(t) = 1 + \frac{c}{\sqrt{t^2 + 1}}. \]

3. Verify that the functions \( \cos(\ln t), \sin(\ln t) \) form a fundamental set of solutions of the differential equation
\[ t^2 y'' + ty' + y = 0 \]
on the interval \((0, +\infty)\). Write down the general solution of the equation.

First, by plugging the functions into the equation we check that they are solutions. For instance for \( y_1(t) = \cos(\ln t) \) we have

\[ t^2 y_1''(t) + t y_1'(t) + y_1(t) = t^2(-\frac{1}{t^2} \cos(\ln t) + \frac{1}{t^2} \sin(\ln t)) + t(-\frac{1}{t} \sin(\ln t)) + \cos(\ln t) = 0. \]

Next we check that they are linearly independent.

\[ W[\cos(\ln t), \sin(\ln t)](t) = \frac{1}{t} \neq 0 \quad \text{on} \quad (0, +\infty) \]
so the functions are linearly independent on \((0, +\infty)\) and therefore they form a fundamental set of solutions. The general solution is

\[
y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t).
\]

4. Find the general solution of the system

\[
\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x}.
\]

Draw a phase portrait for the system. Discuss the stability of the origin.

The eigenvalues of the matrix are \(\lambda = 4 \pm i\). The eigenvector corresponding to the eigenvalue \(4 + i\) is

\[
\begin{bmatrix} 1 \\ i - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Therefore the two linearly independent solutions are

\[
\mathbf{x}_1(t) = e^{4t} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right),
\]

\[
\mathbf{x}_2(t) = e^{4t} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t \right).
\]

The general solution is thus

\[
\mathbf{x}(t) = e^{4t} \left( c_1 \begin{bmatrix} \cos t \\ -\sin t - \cos t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix} \right).
\]

Since the real part of the eigenvalues is positive, the origin is an unstable equilibrium. Trajectories are unstable spirals going clockwise.

5. The rate at which a drug disseminates into the bloodstream is governed by the differential equation

\[
\frac{dx}{dt} = r - kx,
\]

where \(r\) and \(k\) are positive constants. The function \(x(t)\) describes the concentration of the drug in the bloodstream at any time \(t\). Find the limiting value of \(x(t)\) as \(t \to \infty\). At what time is the concentration equal to one-half of this limiting value? Assume that \(x(0) = 0\).

Solving the differential equation we get

\[
\int \frac{dx}{r - kx} = \int dt
\]
\[-\frac{1}{k} \ln |r - kx| = t + c\]

\[r - kx = Ce^{-kt}\]

and since \(x(0) = 0\) we get \(C = r\). Therefore

\[x(t) = \frac{r}{k} - \frac{r}{k} e^{-kt}.\]

and then

\[\lim_{t \to \infty} x(t) = \frac{r}{k}.\]

We now need to find \(t\) such that \(x(t) = r/(2k)\) and so we need

\[\frac{r}{k} - \frac{r}{k} e^{-kt} = \frac{r}{2k}.\]

Therefore

\[e^{-kt} = \frac{1}{2}\]

and then \(-kt = \ln(1/2) = -\ln 2\), which gives

\[t = \frac{\ln 2}{k}.\]