MATH 4317. Homework Assignment #1

Solutions to selected problems.

(1F) We will show that

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

Let $x \in (A \setminus B) \cup (B \setminus A)$. This means that either $x \in A$ and $x \notin B$ or $x \in B$ and $x \in A$. It then follows that $x \in A$ or $x \in B$ and x does not belong to both A and B. Therefore $x \in (A \cup B) \setminus (A \cap B)$ and we get $(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B)$.

Let now $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in A$ or $x \in B$ but x does not belong to both A and B. So either $x \in A$ and then $x \notin B$ or $x \in B$ and then $x \notin A$. Therefore $x \in A \setminus B$ or $x \in B \setminus A$ which means that $x \in (A \setminus B) \cup (B \setminus A)$. Therefore

$$(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A).$$

The two inclusions prove the equality of sets.

1K

$x \in E \setminus \bigcap_{j=1}^{n} A_j$ if and only if $x \in E$ and $x \notin \bigcap_{j=1}^{n} A_j$ which happens if and only if $x \in E$ and x does not belong to at least one A_j which happens if and only if $x \in E \setminus A_j$ for at least one j, i.e. if and only if $x \in \bigcup_{j=1}^{n} (E \setminus A_j)$

$x \in E \setminus \bigcup_{j=1}^{n} A_j$ if and only if $x \in E$ and $x \notin \bigcup_{j=1}^{n} A_j$ which happens if and only if $x \in E$ and x does not belong to A_j for all $j = 1, \ldots, n$, which is equivalent to $x \in E \setminus A_j$ for all $j = 1, \ldots, n$, i.e. if and only if $x \in \bigcap_{j=1}^{n} (E \setminus A_j)$.

(2C) D consists of edges of the square with vertices at points $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ in $\mathbb{R}^2$.
D is not a function since if $-1 < x < 1$ then $(x, 1-|x|)$ and $(x, |x|-1)$ are in D. However $1-|x| \neq |x|-1$.

(2H) Let $(a,b) \in f$. Then, since $g(b) = a$, we have $(b,a) \in g$.
Now let $(b,a) \in g$. Then, since $f(a) = b$, we have $(a,b) \in f$
This shows that

$$(b,a) \in g \text{ if and only if } (a,b) \in f.$$

Obviously this also implies that f is an injection as if $(a_1, b) \in f$ and $(a_2, b) \in f$ then we have $(b, a_1) \in g$ and $(b, a_2) \in g$.
But g is a function and so we must have $a_1 = a_2$.
Therefore f^{-1} ~~and~~ exists and

$$f^{-1} = g = \{(b,a) \in B \times A : (a,b) \in f\}.$$

(3H) $g\circ f = \{(a,c)\in A\times C : \text{there exists } b\in B \text{ such that } (a,b)\in f \text{ and } (b,c)\in g\}$

Since for every $a\in A$, $(a,b)\in f$ for some $b\in B$ and $(b,c)\in g$ for some $c\in C$, $(a,c)\in g\circ f$, which implies that $\mathcal{D}(g\circ f) = A$

Now for every $c\in C$ there exists $b\in B$ such that $(b,c)\in g$, and then there exists $a\in A$ such that $(a,b)\in f$. Therefore $(a,c)\in g\circ f$ which implies that $C = \mathcal{R}(g\circ f)$.

It remains to show that $g\circ f$ is an injection.

Let $(a_1, c)\in g\circ f$ and $(a_2, c)\in g\circ f$. Let $b_1\in B$ be such that $(a_1, b_1)\in f$, $(b_1, c)\in g$. Let $b_2\in B$ be such that $(a_2, b_2)\in f$, $(b_2, c)\in g$. Since g is an injection we get $b_1 = b_2$ and then, since f is an injection we obtain $a_1 = a_2$. Therefore $g\circ f$ is an injection.

(3B) Define $f : O \to \mathbb{N}$ by $f(n) = \frac{n+1}{2}$.

f is one-to-one as if $\frac{n+1}{2} = \frac{m+1}{2}$ then $n = m$.

f is onto $\mathbb{N}$ as for $n\in\mathbb{N}$ $f(2n-1) = \frac{(2n-1)+1}{2} = n$.

(4G) Suppose $\left(\frac{m}{n}\right)^2 = 3$, where m, n do not have common integral factors. Then $m^2 = 3n^2$. We claim that then m must be divisible by 3. If not then

$$m = 3k+1 \quad \text{or} \quad m = 3k+2 \quad \text{for some } k = 0, 1, 2, \ldots$$

If $m = 3k+1$ then

$$m^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 \text{ - not divisible by 3}$$

If $m = 3k+2$ then

$$m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 \text{ - not divisible by 3}$$

So in both cases we obtain a contradiction. Therefore we must have $m = 3k$ for some $k = 1, 2, 3, \ldots$.

But then we get $9k^2 = 3n^2$ which implies $3k^2 = n^2$.

The same argument shows that $n = 3p$ for some $p = 1, 2, \ldots$.

Therefore both m and n are divisible by 3, a contradiction.

(4H) Let $r = \frac{m}{n}$. If

$$r + z = \frac{p}{q} \quad \text{then} \quad z = \frac{p}{q} - \frac{m}{n} = p \cdot \frac{1}{q} \cdot n \cdot \frac{1}{n} - m \cdot \frac{1}{n} \cdot q \cdot \frac{1}{q}$$

$$= pn \cdot \frac{1}{q} \cdot \frac{1}{n} - mq \cdot \frac{1}{n} \cdot \frac{1}{q} = pn \cdot \frac{1}{qn} - mq \cdot \frac{1}{qn} = \frac{pn - mq}{qn}$$

which is a contradiction.

If $r \cdot z = \frac{p}{q}$ then $\frac{m}{n} \cdot z = \frac{p}{q}$

Therefore

$$z = \frac{p}{q} \cdot n \cdot \frac{1}{m} = p \cdot \frac{1}{q} \cdot n \cdot \frac{1}{m} = pn \cdot \frac{1}{q} \cdot \frac{1}{m} = \frac{pn}{qm},$$

a contradiction.

(5C) For $n = 1$

$$(1+a)^1 = 1 + a = 1 + 1 \cdot a$$

so the statement is true. Suppose it is true for some $n \in \mathbb{N}$. We need to show it is true for $n+1$. But

$$(1+a)^{n+1} = (1+a)^n (1+a) \underset{\text{inductive assumption}}{\geq} (1+na)(1+a) = 1 + (n+1)a + na^2$$

$$\geq 1 + (n+1)a$$

since $na^2 \geq 0$. By mathematical induction the statement is therefore true for all $n \in \mathbb{N}$.

5F We first prove that $0 < c^n < 1$ for all $n \in \mathbb{N}$.
Since $0 < c < 1$ we get $0 < c = c^1 < 1$ so the inequality is true for $n = 1$. Suppose now that $0 < c^n < 1$ for some n.
Then

$$0 = 0 \cdot c < c^n \cdot c = c^{n+1} < 1 \cdot c = c < 1.$$

By induction we therefore have $0 < c^n < 1$ for all $n \in \mathbb{N}$.
We now need to show that $c^m \leq c^n$ for all $m \geq n$, $m, n \in \mathbb{N}$. We argue by induction on n.
Since for all $m \in \mathbb{N}$ $c^m < 1$ we get $c^{m+1} \leq c$ for all $m \in \mathbb{N}$, which implies $c^m \leq c$ for all $m \geq 2$. Since $c^1 = c$ we get

$$c^m \leq c$$

for all $m \geq 1$, i.e. the statement is true for $n = 1$.
Suppose now that

$$c^m \leq c^n$$

for some $n \in \mathbb{N}$ and all $m \geq n$. We will show that $c^m \leq c^{n+1}$ for all $m \geq n+1$.
If not then $c^m > c^{n+1}$ for some $m \geq n+1$. But then $c^{m-1} > c^n$. Since $m-1 \geq n$ this contradicts the inductive assumption. Therefore the result follows.

6B Let u be an upper bound of S and $u \in S$.
Let v be another upper bound of S. Since $u \in S$ this implies $u \leq v$. Therefore u is the least upper bound of S and so $u = \sup S$.

6C Let $S = \left\{ \frac{p}{q} \in \mathbb{Q} : \; 0 < \frac{p}{q} < \sqrt{2} \right\}$

S is bounded. Also $\sup S \leq \sqrt{2}$.

If $\sup S = x < \sqrt{2}$ then there exists a rational number $\frac{p}{q}$ such that $x < \frac{p}{q} < \sqrt{2}$. Therefore $\frac{p}{q} \in S$ and $\frac{p}{q} > \sup S$, a contradiction. Therefore $\sup S = \sqrt{2}$.

7F Since $\frac{1}{n+1} < \frac{1}{n}$ we have $I_{n+1} = (0, \frac{1}{n+1}) \subseteq (0, \frac{1}{n}) = I_n$ ~~so the I_n are nested.~~ the sequence (I_n) is nested.

Suppose $x \in \bigcap_{n=1}^{\infty} I_n$. Then $x > 0$. Take $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$. Then $x \notin (0, \frac{1}{n}) = I_n$, a contradiction. So

$$\bigcap_{n=1}^{\infty} I_n = \phi.$$

7G $I_2 \subseteq I_1$ so $[a_2, b_2] \subseteq [a_1, b_1]$ so $a_1 \leq a_2 \leq b_2 \leq b_1$.

Suppose that

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq b_n \leq \ldots \leq b_2 \leq b_1$$

for some $n \in \mathbb{N}$. Since $I_{n+1} \subseteq I_n$ then $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ and so we obtain

$$a_1 \leq a_2 \leq \ldots \leq a_{n+1} \leq b_{n+1} \leq \ldots \leq b_2 \leq b_1.$$

By induction we thus have

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$$
$$b_1 \geq b_2 \geq \ldots, \geq b_n \geq \ldots$$

In the proof of Theorem 7.3 (Nested Cell Property) it is shown that $\xi \leq b_n$ for all $n \in \mathbb{N}$. The same argument also proves that $\eta \geq a_n$ for all $n \in \mathbb{N}$.

Therefore

$$a_1 \le a_2 \le \dots \le a_n \le \dots \le \zeta \le \eta \le \dots \le b_n \le \dots \le b_2 \le b_1.$$

($\zeta \le \eta$ follow from the fact that ζ is a lower bound ~~for~~ $\{b_n\}$).

Let now $x \in [\zeta, \eta]$. Then $a_n \le x \le b_n$ for all $n \in \mathbb{N}$ by the above inequality. Therefore $x \in \bigcap_{n=1}^{\infty} I_n$ and so

$$[\zeta, \eta] \subset \bigcap_{n=1}^{\infty} I_n.$$

Let now $x \in \bigcap_{n=1}^{\infty} I_n$. Then $a_n \le x \le b_n$ for all $n \in \mathbb{N}$. So x is a lower bound of $\{b_n\}$ and x is an upper bound of $\{a_n\}$. Therefore $\zeta \le x \le \eta$ and this yields

$$\bigcap_{n=1}^{\infty} I_n \subset [\zeta, \eta],$$

and we are done.

(3F) Let A be an infinite set and $\{a_1, a_2, \dots\}$ its denumerable subset. Define

$$f(x) = \begin{cases} a_{i+1} & \text{if } x = a_i,\ i = 1, 2, \dots \\ x & \text{if } x \in A \setminus \{a_1, a_2, \dots\}. \end{cases}$$

Then f is one to one and $R(f) = A \setminus \{a_1\}$ which is a proper subset of A.

(3I) Proof by induction. The statement is obviously true if $n = 1$ and if $n = 2$ and $m = 1$. Suppose that for $n \in \mathbb{N}$ there is no bijection between S_n and S_m if $m < n$. We will prove that there is no bijection between S_{n+1} and S_m if $m < n+1$. Suppose there exists $f: S_{n+1} \to S_m$ which is one-to-one and onto. Define $g: S_m \to S_m$ by

$$g(i) = \begin{cases} m & \text{if } i = f(n+1) \\ f(n+1) & \text{if } i = m \\ i & \text{if } i \in S_m \setminus \{m, f(n+1)\}. \end{cases}$$

Since g is one-to-one and onto, the composition $h = g \circ f : S_{n+1} \to S_m$ is one-to-one and onto. But $h(n+1) = m$. Therefore the restriction of h to S_n is one-to-one and onto S_{m-1}. This contradicts the induction assumption.