HOMEWORK ASSIGNMENT #2

(8D) (i) Let $x=(x_1,x_2)$. Then

$$x\cdot x = x_1^2 w_1 + x_2^2 w_2 \geq 0$$

Also if $x\cdot x = x_1^2 w_1 + x_2^2 w_2 = 0$ then $x_1^2 w_1 = x_2^2 w_2 = 0$. Since $w_1, w_2 > 0$ we then get $x_1^2 = x_2^2$ which implies $x_1 = x_2 = 0$.

(ii) Let $x=(x_1,x_2)$, $y=(y_1,y_2)$.

$$x\cdot y = x_1 y_1 w_1 + x_2 y_2 w_2 = y\cdot x$$

(iii) Let $x=(x_1,x_2)$, $y=(y_1,y_2)$, $z=(z_1,z_2)$, $\alpha,\beta\in\mathbb{R}$. Then

$$(\alpha x+\beta y)\cdot z = (\alpha x_1+\beta y_1, \alpha x_2+\beta y_2)\cdot(z_1,z_2)$$

$$= \alpha(x_1 z_1 w_1 + x_2 z_2 w_2) + \beta(y_1 z_1 w_1 + y_2 z_2 w_2) = \alpha(x\cdot z)+\beta(y\cdot z)$$

Therefore $(x_1,x_2)\cdot(y_1,y_2) = x_1 y_1 w_1 + x_2 y_2 w_2$ defines an inner product in $\mathbb{R}^2$. It can be generalized to $\mathbb{R}^p$ as follows. Let $w_1,\dots,w_p > 0$. Then

$$(x_1,\dots,x_p)\cdot(y_1,\dots,y_p) = \sum_{i=1}^{p} x_i y_i w_i.$$

(8P) $\|x+y\|^2 = (x+y)\cdot(x+y) = \|x\|^2 + 2x\cdot y + \|y\|^2$

Therefore $\|x+y\|^2 = \|x\|^2+\|y\|^2$ if and only if $x\cdot y = 0$.

(9D) <u>Interior points</u>: Every $x\in(0,1)$ is an interior point since if $0<x<1$ then $(x-\varepsilon, x+\varepsilon)\subseteq[0,1)$ for $\varepsilon=\min(1-x,x)$.

<u>Boundary points</u>: 0 and 1 are boundary points since every interval $(-\varepsilon,\varepsilon)$ and $(1-\varepsilon,1+\varepsilon)$ for $\varepsilon>0$ contains points in $[0,1)$ and points in $\mathbb{R}\setminus[0,1)$.

Exterior Points: Every point in $\mathbb{R} \setminus [0,1]$ is an exterior point.

If $x < 0$ then $(x-\varepsilon, x+\varepsilon) \cap [0,1) = \emptyset$ for $\varepsilon = -x$.

If $x > 1$ then $(x-\varepsilon, x+\varepsilon) \cap [0,1) = \emptyset$ for $\varepsilon = x-1$.

9 G Let $A \subseteq \mathbb{R}^d$ ~~be open~~. If A is the union of a countable collection of open balls then A is open.

In the other direction if $A \neq \emptyset$ is open, let $\{x_n : n \in \mathbb{N}\}$ be enumeration of all points in A with rational coordinates. (We will show later that this set is denumerable.) For each x_n we denote

$$r_n = \sup\{r : B_r(x_n) \subseteq A\}.$$

If for some n, $\{r : B_r(x_n) \subseteq A\}$ is not bounded then $A = \mathbb{R}^d$ and we are done as then $A = \bigcup_{k=1}^{\infty} B_k(x_n)$.

Therefore assume that $r_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. Since A is open we have $r_n > 0$ for every n. Also

$$B_{r_n}(x_n) = \bigcup_{0<r<r_n} B_r(x_n) \subseteq A \quad \text{for every } n.$$

We claim that $\bigcup_{n \in \mathbb{N}} B_{r_n}(x_n) = A$.

Let then $x \in A$. Since A is open $B_r(x) \subseteq A$ for some $r > 0$.

If $x = (x_1, \ldots, x_d)$ we choose for every $i = 1, \ldots, d$ a rational number y_i such that

$$x_i - \frac{r}{2\sqrt{d}} < y_i < x_i + \frac{r}{2\sqrt{d}}.$$

Denote $y = (y_1, \ldots, y_d)$. Then

$$\|y-x\| = \sqrt{|y_1-x_1|^2+\dots+|y_d-x_d|^2} < \sqrt{d\,\frac{r^2}{4d}} = \frac{r}{2}.$$

Therefore $y \in A$ and so $y = x_m$ for some $m \in \mathbb{N}$. (This shows in particular that the set of all points in A with rational coordinates is non-empty. Repeating the above argument it is easy to see that this set is infinite and so it must be denumerable as it is a subset of $\mathbb{Q}^d$.)

Let now $z \in \mathbb{R}^d$ be such that $\|z - x_m\| < \frac{r}{2}$. Then

$$\|x-z\| \le \|x-x_m\| + \|x_m - z\| < \frac{r}{2} + \frac{r}{2} = r.$$

Therefore $z \in A$ and so $B_{\frac{r}{2}}(x_m) \subseteq A$, which implies that $r_m \geq \frac{r}{2}$, and so $x \in B_{r_m}(x_m)$. This shows that

$$A \subseteq \bigcup_{n\in\mathbb{N}} B_{r_n}(x_n).$$

Since we already know that $B_{r_n}(x_n) \subseteq A$ for every n we thus obtain $A = \bigcup_{n\in\mathbb{N}} B_{r_n}(x_n)$.

10 D Since for every $n \in \mathbb{N}$ and $\varepsilon > 0$, $\left(\frac{1}{n}-\varepsilon, \frac{1}{n}+\varepsilon\right)$ contains a point in A (namely $\frac{1}{n}$) and a point in $\mathbb{R}\setminus A$, every point in A is a boundary point of A.

Let $\varepsilon > 0$. Since $(-\varepsilon, \varepsilon) \cap A = \left\{\frac{1}{n} : n > \frac{1}{\varepsilon}\right\} \neq \emptyset$, every neighborhood of 0 contains points in A different from 0. Therefore 0 is a cluster point of A. There are no other cluster points.

If $x < 0$ then $(-\infty, 0) \cap A = \emptyset$ and $(-\infty, 0)$ is a neighborhood of x so x cannot be an accumulation point of A.

If $0 < x \leq 1$ let $n \in \mathbb{N}$ be the natural number such that
$$\frac{1}{n+1} < x \leq \frac{1}{n}.$$
If $x = \frac{1}{n}$ then $\left(\frac{1}{n+1}, \frac{1}{n-1}\right) \cap (A \setminus \{x\}) = \phi$ if $n > 1$ and
$\left(\frac{1}{2}, +\infty\right) \cap (A \setminus \{x\}) = \phi$ if $n = 1$.
If $\frac{1}{n+1} < x < \frac{1}{n}$ then $\left(\frac{1}{n+1}, \frac{1}{n}\right) \cap A = \phi$.
If $x > 1$ then $(1, +\infty) \cap A = \phi$.
Therefore none of the points $x > 0$ can be a cluster point of A as in each of the above cases we have found a neighborhood of x that ~~that~~ is either disjoint from A or it does not contain any other points of A different from x.

10 E If x is a cluster point of $A \cap B$ then for every neighborhood U of x there exists a point $y \neq x$ such that
$$y \in U \cap (A \cap B) = (U \cap A) \cap (U \cap B)$$
Therefore for every neighborhood U of x both $U \cap A$ and $U \cap B$ contain a point different from x. This shows that x is a cluster point of both A and B.

11 A Denote $G_n = \left\{(x,y) : x^2 + y^2 < 1 - \frac{1}{n}\right\}$ for $n \geq 2$. Then
$$A = \{(x,y) : x^2+y^2 < 1\} = \bigcup_{n \geq 2} G_n.$$
However A cannot be the union of a finite number of G_n. If
$$A \subseteq G_{n_1} \cup \ldots \cup G_{n_k} \quad \text{for some } n_1, \ldots, n_k \geq 2$$
then, since $G_2 \subseteq G_3 \subseteq \ldots \subseteq G_n \subseteq G_{n+1} \subseteq \ldots$, $A \subseteq G_m$ for $m = \max(n_1, \ldots, n_k)$.
But this means $\{(x,y) : x^2+y^2 < 1\} \subseteq \left\{(x,y) : x^2+y^2 < 1 - \frac{1}{m}\right\}$ which is a contradiction.

11 B

$$\mathbb{R}^2 = \bigcup_{n \in \mathbb{N}} B_n((0,0)).$$

Since the sets $B_n((0,0))$ create a nested family, if

$$\mathbb{R}^2 = B_{n_1}((0,0)) \cup \dots \cup B_{n_k}((0,0))$$

for some $n_1, \dots, n_k \in \mathbb{N}$, we then would have

$$\mathbb{R}^2 = B_m((0,0)), \quad \text{where } m = \max(n_1, \dots, n_k).$$

However this is an obvious contradiction and so $\mathbb{R}^2$ is not compact.

12 A (a)(i) $A = \{x = (x_1, \dots, x_d) : x_1 < 0\}$, $B = \{x : x_1 < 0\}$

A and B are connected but $A \cup B$ is not since A and B are open and disjoint.

(ii) A - as above, $B = \{x : x_1 \geq 0\}$. $A \cup B = \mathbb{R}^d$ which is connected.

(b) (i) $A = \{x : x_1 > 0\}$, $B = \{x : x_1 < 1\}$. A, B are connected.

$$A \cap B = \{x : 0 < x_1 < 1\} \text{ which is connected}$$

(ii) $A = \mathbb{R}^d \setminus \{x : x_1 \leq 0, x_2 \geq 0\}$, $B = \mathbb{R}^d \setminus \{x : x_1 \geq 0, x_2 \leq 0\}$

A, B are connected (since they are pathwise connected) but

$$A \cap B = \{x : x_1, x_2 > 0 \text{ or } x_1, x_2 < 0\} = \{x : x_1, x_2 < 0\} \cup \{x : x_1, x_2 > 0\}.$$

So $A \cap B$ is the union of two open, disjoint sets and ~~and~~ therefore it is not connected.

(c) (i) ~~A\B~~ Let A and B be as in (b)(i). Then

$$A \setminus B = \{x : x_1 \geq 1\} \text{ which is connected.}$$

(c) (ii) $A = \mathbb{R}^d$, $B = \{x : 0 \le x_1 \le 1\}$

A and B are connected but

$$A \setminus B = \{x : x_1 < 0\} \cup \{x : x_1 > 1\}$$

is disconnected.

12 C Suppose there exist open sets A, B such that $C^- \cap A$ and $C^- \cap B$ are disjoint, non-empty, and

$$(C^- \cap A) \cup (C^- \cap B) = C^-.$$

Since $C \subseteq C^-$ we obviously have that $C \cap A$ and $C \cap B$ are disjoint. Also

$$(C \cap A) \cup (C \cap B) = C \cap (A \cup B) = C \cap C^- \cap (A \cup B)$$

$$= C \cap \left((C^- \cap A) \cup (C^- \cap B)\right) = C \cap C^- = C.$$

We will show that $C \cap A$ and $C \cap B$ are non-empty. Let us show it for $C \cap A$. If $C \cap A$ $\emptyset$ then $C \subseteq \mathbb{R}^p \setminus A$. But $\mathbb{R}^p \setminus A$ is closed and then by the definition of C^- it follows that $C^- \subseteq \mathbb{R}^p \setminus A$, which implies that $C^- \cap A \neq \emptyset$ which is a contradiction.

Therefore we have shown that $C \cap A$ and $C \cap B$ are disjoint, non-empty, and $(C \cap A) \cup (C \cap B) = C$. This contradicts the connectedness of C.