HOMEWORK ASSIGNMENT # 3

14 I Let $\lim \frac{x_{n+1}}{x_n} = a < 1$. Set $\varepsilon_1 = \frac{1-a}{2}$, $r = \frac{1+a}{2} < 1$.

There exists $K(\varepsilon_1) \in \mathbb{N}$ such that for $n \geq K(\varepsilon_1)$

$$\frac{x_{n+1}}{x_n} < a + \varepsilon_1 = \frac{a+1}{2} = r.$$

Therefore for $n > K(\varepsilon_1)$, by induction,

$$0 < x_n = x_{K(\varepsilon_1)} \frac{x_{K(\varepsilon_1)+1}}{x_{K(\varepsilon_1)}} \cdot \ldots \cdot \frac{x_n}{x_{n-1}} < x_{K(\varepsilon_1)} r^{n-K(\varepsilon_1)}$$

$$= \frac{x_{K(\varepsilon_1)}}{r^{K(\varepsilon_1)}} r^n = C r^n \quad \text{for } C = \frac{x_{K(\varepsilon_1)}}{r^{K(\varepsilon_1)}}$$

Since $r^n \to 0$ for $\varepsilon > 0$ there is $K(\varepsilon)$ such that for $n \geq K(\varepsilon)$ $\quad 0 < r^n < \frac{\varepsilon}{C}$.

Then for $n > \max(K(\varepsilon_1), K(\varepsilon))$

$$|x_n| < C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

Therefore $x_n \to 0$.

14 H If $x_n = n b^n$ then $\frac{x_{n+1}}{x_n} = \frac{n+1}{n} b = b + \frac{b}{n} \to b < 1$

Therefore by problem 14 I we get $\lim(n b^n) = 0$.

Another proof: Using that $n^{\frac{1}{n}} \to 1$ we get that $n^{\frac{1}{n}} b \to b < 1$ so there exists a number $K \in \mathbb{N}$ such that for $n \geq K$ $0 < n^{\frac{1}{n}} b < \frac{1+b}{2} < 1$. Therefore $|n b^n| < \left(\frac{1+b}{2}\right)^n$ for $n \geq K$.

Since $\left(\frac{1+b}{2}\right)^n \to 0$ we get that $nb^n \to 0$ by Theorem 14.9.

(14 K) Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ and $\frac{x_{n+1}}{x_n} = \frac{n}{n+1} = 1 - \frac{1}{n+1} \to 1$

Let $x_n = n$. Then (x_n) is divergent but $\frac{x_{n+1}}{x_n} = 1 + \frac{1}{n} \to 1$.

(15 E) Let (x_n) and (y_n) be any divergent sequences in $\mathbb{R}$. Denote $X = ((x_n, 0))$, $Y = ((0, y_n))$. Then $X \cdot Y = ((0,0))$ and is convergent but ~~neither~~ X ~~nor~~ and Y are divergent.

(15 L) Since $0 < a \le b$, $a^n \le b^n$ for every $n \in \mathbb{N}$.

Therefore $b^n \le a^n + b^n \le 2b^n$

and then it follows that

$$b \le (a^n + b^n)^{\frac{1}{n}} \le 2^{\frac{1}{n}} b$$

Since $2^{\frac{1}{n}} \to 1$ we must have $(a^n + b^n)^{\frac{1}{n}} \to b$ (use 15 B or prove it directly).

(16 G) $0 < x_n = \frac{1}{n+1} + \dots + \frac{1}{2n} < \frac{n}{n+1} < 1$.

Therefore (x_n) is bounded. Also

$$x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0.$$

Therefore (x_n) is monotone increasing. Thus, by the Monotone Convergence Theorem, the sequence (x_n) is convergent.

16 M (a)

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) \to e\cdot 1 = e$$

(b)

$$\left(1+\frac{1}{2n}\right)^{n} = \left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{\frac{1}{2}} \to e^{\frac{1}{2}} = \sqrt{e}$$

(c)

$$\frac{\left(1+\frac{2}{n}\right)^{n}}{\left(1+\frac{1}{n}\right)^{n}} = \left(\frac{n+2}{n+1}\right)^{n} = \left(1+\frac{1}{n+1}\right)^{n} = \left(1+\frac{1}{n+1}\right)^{n+1}\frac{n+1}{n+2} \to e$$

$$\therefore \left(1+\frac{2}{n}\right)^{n} \to e^{2}$$

17 B Let $x>0$, and let $\varepsilon>0$. Take $K(\varepsilon)\in\mathbb{N}$ such that $K(\varepsilon) > \max\left(\frac{1}{x}, \frac{1}{\varepsilon x}\right)$. Then for $n \geq K(\varepsilon)$ $\frac{1}{n} \leq \frac{1}{K(\varepsilon)} < x$ and so $g_n(x) = \frac{1}{nx}$. Therefore for $n \geq K(\varepsilon)$

$$|g_n(x) - 0| = \frac{1}{nx} \leq \frac{1}{K(\varepsilon)}\cdot\frac{1}{x} < \varepsilon x \cdot \frac{1}{x} = \varepsilon$$

which shows that $\lim g_n(x) = 0$ for all $x>0$.

17M $f_n(x) \to 0$ for all $x \in \mathbb{R}$.

However convergence is not uniform as

$$\left| f_n\left(\tfrac{1}{n}\right) - 0 \right| = \frac{1}{1+1} = \frac{1}{2}.$$

18C Let $\varepsilon > 0$ and denote $x^* = \liminf x_n$, $y^* = \liminf y_n$.

By the definition of x^* there must exist w such that $x^* - \varepsilon \leq w \leq x^*$ and such that there are at most finitely many n such that $x_n < w$. Therefore there are at most finitely many n such that $x_n < x^* - \varepsilon$. Likewise we get that there are at most finitely many n such that $y_n < y^* - \varepsilon$. Therefore there are at most finitely many n such that

$$x_n + y_n < x^* + y^* - 2\varepsilon.$$

This implies that

$$x^* + y^* - 2\varepsilon \leq \liminf (x_n + y_n).$$

Since $\varepsilon > 0$ is arbitrary we thus obtain

$$x^* + y^* \leq \liminf (x_n + y_n).$$

18 F If $\limsup \left(\frac{x_{n+1}}{x_n}\right) = +\infty$ there is nothing to prove.

Let then $\limsup \left(\frac{x_{n+1}}{x_n}\right) = L \geq 0$, and let $\varepsilon > 0$.

Since there are at most finitely many n such that

$$\frac{x_{n+1}}{x_n} > L + \varepsilon$$

there exists $K(\varepsilon) \in \mathbb{N}$ such that for all $n \geq K(\varepsilon)$

$$\frac{x_{n+1}}{x_n} \leq L + \varepsilon.$$

But then for $n > K(\varepsilon)$

$$x_n = \frac{x_n}{x_{n-1}} \cdot \ldots \cdot \frac{x_{K(\varepsilon)+1}}{x_{K(\varepsilon)}} x_{K(\varepsilon)} \leq (L+\varepsilon)^{n-K(\varepsilon)} x_{K(\varepsilon)}$$

$$= (L+\varepsilon)^n \frac{x_{K(\varepsilon)}}{(L+\varepsilon)^{K(\varepsilon)}}.$$

Let $K_1(\varepsilon) \in \mathbb{N}$ be such that

$$\left(\frac{x_{K(\varepsilon)}}{(L+\varepsilon)^{K(\varepsilon)}}\right)^{\frac{1}{n}} < 1 + \frac{\varepsilon}{L+\varepsilon} \quad \text{for } n \geq K_1(\varepsilon).$$

Then for $n > \max(K(\varepsilon), K_1(\varepsilon))$ we have

$$x_n^{\frac{1}{n}} \leq (L+\varepsilon)\left(1 + \frac{\varepsilon}{L+\varepsilon}\right) = L + 2\varepsilon.$$

Therefore there are at most finitely many n such that

$$x_n^{\frac{1}{n}} > L + 2\varepsilon.$$

This implies that

$$\limsup \left(x_n^{\frac{1}{n}}\right) \leq L + 2\varepsilon \quad \text{for all } \varepsilon > 0.$$

Therefore we conclude that

$$\limsup \left(x_n^{\frac{1}{n}}\right) \leq L.$$