HOMEWORK ASSIGNMENT #4

20B $f(x)$ is the sum of functions that are products of continuous functions (e.g. $a_n x^n = a_n \cdot \underbrace{x \cdot \ldots \cdot x}_{n\text{-times}}$) so it is continuous. However we will show by definition that $h(x) = a_k x^k$ is continuous on $\mathbb{R}$.

Let $a \in \mathbb{R}$, and let $\varepsilon > 0$. If $|x-a| < 1$ then $|x| < |a|+1$. Now if $|x-a| < \min\left(1, \frac{k(|a|+1)^{k-1}}{|a_k|}\right)$ then

$$|h(x)-h(a)| = |a_k|\,|x^k - a^k| = |a_k|\,|x-a|\,|x^{k-1} + x^{k-2}a + \ldots + x a^{k-2} + a^{k-1}|$$

$$\leq |a_k|\,|x-a|\,k\left(|a|+1\right)^{k-1} < \varepsilon .$$

20E Let $\varepsilon > 0$. Take $\delta = \varepsilon$. Then if $|x - \frac{1}{2}| < \delta$ we have

$$f(x) - f(\tfrac{1}{2}) = \begin{cases} x - \frac{1}{2} & x \text{ is irrational} \\ 1 - x - \frac{1}{2} = \frac{1}{2} - x & x \text{ is rational} \end{cases}$$

so $|f(x) - f(\frac{1}{2})| = |x - \frac{1}{2}| < \varepsilon \implies f$ is continuous at $x = \frac{1}{2}$.

Let now $x \in \mathbb{Q}$, $x \neq \frac{1}{2}$. Let $x_n \in \mathbb{R} \setminus \mathbb{Q}$, $x_n \to x$. Then

$f(x_n) = 1 - x_n \to 1 - x \neq x = f(x)$. Therefore f is discontinuous at such x.

If $x \in \mathbb{R} \setminus \mathbb{Q}$ then we take $x_n \in \mathbb{Q}$, $x_n \to x$. Then

$f(x_n) = x_n \to x \neq 1 - x = f(x)$. Therefore f is discontinuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$.

(21L) (i) $\|0\|_{pq} = 0$

If $\|f\|_{pq} = 0$ then $\|f(x)\| = 0$ for every $x \in \mathbb{R}^p$, $\|x\| \le 1$. Now if $x \in \mathbb{R}^p$ then $\left\| \frac{x}{\|x\|} \right\| \le 1$ so $f(x) = f\left(\frac{x}{\|x\|}\|x\|\right) = \|x\| f\left(\frac{x}{\|x\|}\right) = 0$. Therefore $f = 0$.

(ii) $\|af\|_{pq} = \sup\{\|af(x)\| : x \in \mathbb{R}^p, \|x\| \le 1\} = \sup\{|a|\|f(x)\| : x \in \mathbb{R}^p, \|x\| \le 1\}$
$= |a| \sup\{\|f(x)\| : x \in \mathbb{R}^p, \|x\| \le 1\} = |a| \|f\|_{pq}$.

(iii) Since

$$\|f(x) + g(x)\| \le \|f(x)\| + \|g(x)\|$$

$$\|f+g\|_{pq} = \sup\{\|f(x)+g(x)\| : x \in \mathbb{R}^p, \|x\| \le 1\} \le \sup\{\|f(x)\| + \|g(x)\| : x \in \mathbb{R}^p, \|x\| \le 1\}$$
$$\le \sup\{\|f(x)\| : x \in \mathbb{R}^p, \|x\| \le 1\} + \sup\{\|g(x)\| : x \in \mathbb{R}^p, \|x\| \le 1\} = \|f\|_{pq} + \|g\|_{pq}.$$

Now if $x = 0$ then $\|f(0)\| = 0 \le \|f\|_{p,q} \cdot 0$. If $x \ne 0$ then

$$\|f(x)\| = \left\| f\left(\|x\| \frac{x}{\|x\|}\right) \right\| = \|x\| \left\| f\left(\frac{x}{\|x\|}\right) \right\|$$

$$\le \|x\| \sup\{\|f(y)\| : y \in \mathbb{R}^p, \|y\| \le 1\} = \|x\| \|f\|_{pq}.$$

(22F) Let $f : D \to \mathbb{R}$ be continuous and such that $f(D) = \{0,1\}$. Let $A = (-\infty, \frac{1}{2})$, $B = (\frac{1}{2}, +\infty)$. Let A_1, B_1 be open sets in $\mathbb{R}^p$ such that $A_1 \cap D = f^{-1}(A)$, $B_1 \cap D = f^{-1}(B)$. Then $A_1 \cap D$ and $B_1 \cap D$ are disjoint and $(A_1 \cap D) \cup (B_1 \cap D) = f^{-1}(A \cup B) = D$. Therefore A_1, B_1 form a disconnection of D.

Different proof: If D were connected then $f(D)$ would be connected but $\{0,1\}$ is not connected.

Let now D be disconnected, i.e. suppose that $D = (A \cap D) \cup (B \cap D)$, where A, B are open, $A \cap B$ and $B \cap D$ are non-empty and disjoint.

Define $f : D \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \cap D \\ 1 & \text{if } x \in B \cap D \end{cases}$$

Let $G \subseteq \mathbb{R}$ be open. Then

$$f^{-1}(G) = \begin{cases} \phi & \text{if } \{0,1\} \cap G = \phi \\ A \cap D & \text{if } 0 \in G \text{ and } 1 \notin G \\ B \cap D & \text{if } 0 \notin G \text{ and } 1 \in G \\ D & \text{if } \{0,1\} \subseteq G. \end{cases}$$

So in each case $f^{-1}(G) = D \cap G_1$ for some open subset G_1 of $\mathbb{R}^p$. Thus f is continuous.

22K Example 1: $f(x) = \arctan x$

Example 2:

$$f(x) = \begin{cases} \dfrac{x^2}{1+x^2} & x \geq 0 \\ -\dfrac{x^2}{1+x^2} & x < 0 \end{cases}$$

23C Let I be a ~~…~~ closed cell such that $B \subseteq I$. Divide I into a finite number of ~~…~~ closed cells whose sides have length less than $\frac{\delta}{\sqrt{p}}$, where δ is such that if $\|x-y\| < \delta$, $x, y \in B$, then $\|f(x) - f(y)\| < 1$. Let $I_1, \ldots, I_n$ be all the cells ~~such that I~~ whose intersection with B is non-empty. (Notice that then $B \subset \bigcup_{i=1}^{n} I_i$.) We now choose a point ~~…~~ $x_i \in B \cap I_i$ for $i = 1, \ldots, n$. Denote

$$M = \max\{\|f(x_i)\| + 1 : i = 1, \ldots, n\}.$$

Let now $x \in B$. Therefore $x \in B \cap I_i$ for some $i = 1, \ldots, n$.
But then

$$\|f(x)\| \leq \|f(x_i)\| + \|f(x) - f(x_i)\| \leq M$$

since $\|x - x_i\| < \delta$ as ~~xi~~ x and x_i are both in I_i.
Therefore f is bounded.
If B is not bounded then for instance $f(x) = x$ is uniformly continuous from B to B but is not bounded.

23G Let $M = \sup\{\|f(x)\| : 0 \leq x \leq p\}$. ($[0,p]$ is compact.)
If $x \in \mathbb{R}$ there exists $k \in \mathbb{Z}$ such that $0 \leq x + kp \leq p$.
Therefore $\|f(x)\| = \|f(x+kp)\| \leq M$, so f is bounded.
Since f is continuous and $[0, 2p]$ is compact, f is uniformly continuous on $[0, 2p]$. Let $\varepsilon > 0$ and let $\delta(\varepsilon) > 0$ be such that if $|x - y| < \delta$, $x, y \in [0, 2p]$, then $\|f(x) - f(y)\| < \varepsilon$.
Set $\delta_1 = \min(\delta, p)$. Suppose now that $x, y \in \mathbb{R}$, $|x - y| < \delta_1$.
We can assume that $x \leq y$. Let $k \in \mathbb{Z}$ be such that $0 \leq x + kp < p$. Then $0 \leq y + kp < 2p$ and
$|x + kp - (y + kp)| = |x - y| < \delta_1 \leq \delta$. Therefore
$\|f(x) - f(y)\| = \|f(x+kp) - f(y+kp)\| < \varepsilon$.
This proves uniform continuity of f.

24E(6) If $0 \leq x < 1$ then $\left|\frac{x^n}{1+x^n}\right| \leq |x|^n \to 0$
If $x > 1$ then $\left|\frac{x^n}{1+x^n} - 1\right| = \frac{1}{1+x^n} \to 0$
Therefore

the sequence (f_n) converges to

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & x > 1 \end{cases}$$

Since f is not continuous convergence cannot be uniform.
Alternatively one can directly check that

$$\frac{\left[\left(\frac{1}{2}\right)^{\frac{1}{n}}\right]^n}{1+\left[\left(\frac{1}{2}\right)^{\frac{1}{n}}\right]^n} = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3} \not\to 0$$

$$\frac{\left[2^{\frac{1}{n}}\right]^n}{1+\left[2^{\frac{1}{n}}\right]^n} = \frac{2}{1+2} = \frac{2}{3} \not\to 1$$

However convergence is uniform on every interval $[0,r]$, $0<r<1$, since

$$\left\| \frac{x^n}{1+x^n} \right\|_{[0,r]} \leq \| x^n \|_{[0,r]} = r^n \to 0 \quad \text{as } n \to \infty.$$

Also convergence is uniform on every interval $[r,+\infty)$, $r>1$, since

$$\left\| \frac{x^n}{1+x^n} - 1 \right\|_{[r,+\infty)} = \left\| \frac{1}{1+x^n} \right\|_{[r,+\infty)} \leq \frac{1}{1+r^n} \to 0 \quad \text{as } n \to \infty.$$

24] Since f is continuous on $I=[0,1]$, f is uniformly continuous. Let $\varepsilon>0$. Let $k \in \mathbb{N}$ be such that if $|x-y| \leq \frac{1}{k}$, $x,y \in I$, then $|f(x)-f(y)| < \varepsilon/2$.

Denote $x_i := \frac{i}{k}$ for $i = 0, 1, \dots, k$. Let $M \in \mathbb{N}$ be such that if $n \geq M$ then

$$|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{2} \quad \text{for all } i = 0, 1, \dots, k.$$

Now if $x \in I$, then $x_i \leq x \leq x_{i+1}$ for some $i = 0, 1, \dots, k-1$.

Therefore

$$f_n(x) \leq f_n(x_{i+1}) < f(x_{i+1}) + \frac{\varepsilon}{2} < f(x) + \varepsilon$$

and

$$f_n(x) \geq f_n(x_i) > f(x_i) - \frac{\varepsilon}{2} > f(x) - \varepsilon$$

Thus we obtain that if $n \geq M$ then for every $x \in I$

$$|f_n(x) - f(x)| < \varepsilon$$

which shows uniform convergence

260 Since $\mathcal{F}$ is bounded, f^* is well defined. For every $\varepsilon > 0$ let $\delta(\varepsilon) > 0$ be such that if $x, y \in D$, $\|x - y\| < \delta$, and $g \in \mathcal{F}$, then $|g(x) - g(y)| < \varepsilon$. We will show that the same $\delta(\varepsilon)$ works for f^*.

Let $x, y \in D$, $\|x - y\| < \delta$. For every $n \in \mathbb{N}$ let $g_n \in \mathcal{F}$ be such that

$$f^*(x) - \frac{1}{n} \leq g_n(x) \leq f^*(x).$$

Then $f^*(x) - \frac{1}{n} \leq g_n(x) \leq g_n(y) + \varepsilon \leq f^*(y) + \varepsilon$

Letting $n \to +\infty$ we thus get $f^*(x) \leq f^*(y) + \varepsilon$

Replacing the roles of x and y we also obtain a symmetric estimate $f^*(y) \leq f^*(x) + \varepsilon$. Together they show that

$$|f^*(x) - f^*(y)| \leq \varepsilon.$$