Math. 4317, Practice Test 1

1. Let $a > 2$ and $b = 1 + \sqrt{a-1}$. Show that $2 < b < a$.

2. Show that $2^n + 3^n$ is a multiple of 5 for all odd $n \in \mathbb{N}$.

3. Let $f : X \times Y \to \mathbb{R}$ be a function with bounded range. Show that

$$\sup_{x \in X} \inf_{y \in Y} f(x,y) \leq \inf_{y \in Y} \sup_{x \in X} f(x,y).$$

4. Let $f : A \to B$ be a surjection and $g : B \to C$ be such that $g \circ f$ is an injection. Prove that both f and g are injections.

5. Can you find a compact set $A \subset \mathbb{R}$ and a function $f : A \to A$ such that $f(A)$ is not compact?

1. We first show that $\sqrt{a-1} > 1$. If $\sqrt{a-1} \le 1$ then $a-1 = \sqrt{a-1} \cdot \sqrt{a-1} \le \sqrt{a-1} \cdot 1 \le 1$ which is a contradiction.

Therefore $b = 1 + \sqrt{a-1} > 2$.

If $b \ge a$ then $b - 1 \ge a - 1 > \sqrt{a-1}$ which is a contradiction. Therefore we must have $b < a$.

2. The proof is by induction. $2^1 + 3^1 = 5 = 5 \cdot 1$ so the statement is true for $k = 1$. Suppose that the statement is true for an odd number $2k-1$. We need to show that it is true for $2(k+1)-1$, i.e. that $2^{2k-1} + 3^{2k-1} = 5m$ for some $m \in \mathbb{N}$. We now have

$$2^{2k+1} + 3^{2k+1} = 2^{2k-1}4 + 3^{2k-1}9 = (2^{2k-1} + 3^{2k-1})4 + 3^{2k-1}5 = 5(4m + 3^{2k-1})$$

which proves the claim.

3. For every $x_0 \in x$ and $y_0 \in Y$ we have

$$\inf_{y \in Y} f(x_0, y) \le f(x_0, y_0).$$

Therefore $\sup_{x \in X} f(x, y_0)$ is an upper bound of $\{\inf_{y \in Y} f(x, y) : x \in X\}$ and so

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \le \sup_{x \in X} f(x, y_0).$$

But this implies that $\sup_{x \in X} \inf_{y \in Y} f(x, y)$ is a lower bound of $\{\sup_{x \in X} f(x, y) : y \in Y$ and so we obtain

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \le \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

4. Let $(b, c) \in g$ and $(b', c) \in g$. Since f is a surjection there exist $a, a' \in A$ such that $(a, b) \in f$ and $(a', b') \in f$. But then $(a, c) \in g \circ f$ and $(a', c) \in g \circ f$. Therefore $a = a'$ and this implies $b = b'$ which means that g is an injection.

To show that f is an injection we take $(a, b) \in f$ and $(a', b) \in f$. Let $c \in C$ be such that $(b, c) \in g$. Then $(a, c) \in g \circ f$ and $(a', c) \in g \circ f$ which again implies that $a = a'$ which proves the claim.

5. Take $A = [0, 1]$ and f defined by $f(x) = x/2$ for $0 \le x < 1$ and $f(1) = 1$. Then $f(A) = [0, 1/2) \cup \{1\}$. This set is not closed so it is not compact.