HOMEWORK ASSIGNMENT #1

27.A(f) $H(x) = \frac{1}{x^2}$, $x \neq 0$.

Let $c \neq 0$. Then

$$H'(c) = \lim_{x \to c} \frac{H(x) - H(c)}{x - c} = \lim_{x \to c} \frac{\frac{1}{x^2} - \frac{1}{c^2}}{x - c} = \lim_{x \to c} \frac{c^2 - x^2}{x^2 c^2 (x - c)}$$

$$= \lim_{x \to c} \frac{-(c + x)}{x^2 c^2} = -\frac{2c}{c^4} = -\frac{2}{c^3}.$$

27.H Let $g(x) = f(x) - C(x - a)$. Suppose $A < B$. Since g is differentiable on $[a, b]$, g is continuous on $[a, b]$, and thus there exists $c \in [a, b]$ such that

$$g(c) = \min_{a \leq x \leq b} g(x).$$

Since $g'(a) = f'(a) - C = A - C < 0$, we obtain that $g(x) < g(a)$ for $a < x < a + \delta$ for some $\delta > 0$ (see Lemma 27.3), and since $g'(b) = f'(b) - C = B - C > 0$, $g(x) < g(b)$ for $b - \delta < x < b$ for some $\delta > 0$. Therefore $c \in (a, b)$ and then

$$0 = g'(c) = f'(c) - C.$$

If $A > B$ we apply the same arguments to the function $-g(x)$.

27.L Let $\varepsilon > 0$ and let $\delta(\varepsilon)$ be such that

$$\left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \varepsilon \quad \text{if } 0 < |x-c| < \delta(\varepsilon).$$

Then if $a \le x < c < y \le b$, $0 < y-x < \delta$, we have

$$\left| \frac{f(y)-f(x)}{y-x} - f'(c) \right| \le \left| \frac{y-c}{y-x} \left(\frac{f(y)-f(c)}{y-c} - f'(c) \right) \right|$$

$$+ \left| \frac{c-x}{y-x} \left(\frac{f(c)-f(x)}{x-c} - f'(c) \right) \right|$$

$$\le \frac{y-c}{y-x} \varepsilon + \frac{c-x}{y-x} \varepsilon = \varepsilon$$

Inequality is also true if $x=c$ or $y=c$.

27.S

(a) By Mean Value Theorem, for every $x \in [0,+\infty)$ and $h > 0$, there exist $x < c_x < x+h$ such that

$$\frac{f(x+h)-f(x)}{h} = f'(c_x).$$

Therefore $\lim\limits_{x \to +\infty} c_x = +\infty$ and thus

$$\lim_{x\to+\infty} \frac{f(x+h)-f(x)}{h} = \lim_{x\to+\infty} f'(c_x) = b.$$

(b) $0 = \lim_{n\to+\infty} \left[f(n+1) - f(n) \right] = \lim_{n\to+\infty} f'(c_n)$

where $n \le c_n \le n+1$.

$$\therefore \quad b = \lim_{n\to+\infty} f'(c_n) = 0.$$

(c) Let $\varepsilon > 0$ and let M_ε be such that if $x \geqslant M_\varepsilon$, then

$$|f'(x) - b| \le \varepsilon.$$

Let c_x be such that $M_\varepsilon \le c_x \le x$ and

$$f(x) - f(M_\varepsilon) = f'(c_x)(x - M_\varepsilon).$$

Now

$$\frac{f(x)}{x} = \frac{f(x) - f(M_\varepsilon)}{x} + \frac{f(M_\varepsilon)}{x} = \frac{f'(c_x)(x-M_\varepsilon)}{x} + \frac{f(M_\varepsilon)}{x}$$

$$= f'(c_x) + \frac{f(M_\varepsilon) - f'(c_x) M_\varepsilon}{x}$$

Therefore if $x \geqslant M_\varepsilon$ we have

$$\left| \frac{f(x)}{x} - b \right| \le |f'(c_x) - b| + \frac{|f(M_\varepsilon)| + |f'(c_x)| M_\varepsilon}{x}$$

$$\le \varepsilon + \frac{|f(M_\varepsilon)| + (|b| + \varepsilon) M_\varepsilon}{x}, \text{ which implies}$$

$$\left| \frac{f(x)}{x} - b \right| \le 2\varepsilon \quad \text{if} \quad x \geqslant \max\left(M_\varepsilon, \frac{|f(M_\varepsilon)| + (|b|+\varepsilon) M_\varepsilon}{\varepsilon} \right).$$

28.E Define $f(x) = (1+x)^r - (1+rx)$, $x > -1$.

$f'(x) = r(1+x)^{r-1} - r$.

Therefore $f'(x) > 0$ if and only if $(1+x)^{r-1} > 1$

$\iff (1+x)^{1-r} < 1 \iff -1 < x < 0$.

$f'(0) = 0$

$f'(x) < 0 \iff (1+x)^{1-r} > 1 \iff x > 0$.

Therefore f is ~~de~~increasing on $(-1, 0]$ and f is decreasing on $[0, +\infty)$. Since $f(0) = 0$, this implies $f(x) \le f(0) = 0$ for $-1 < x \le 0$ and $f(x) \le f(0) = 0$ for $x \geqslant 0$. Therefore $f(x) \le 0$ for all $x > -1$ which proves the claim.

28.F Let $x_1, \dots, x_n$ be the roots of p' in (a, b).

Denote by $x_1, \dots, x_k$ the roots where p' changes sign, and by $x_{k+1}, \dots, x_n$ these where p' does not change sign.

Since the number of intervals in (a,b) with nonnegative derivative must be equal to the number of intervals with nonpositive derivative, k must be odd.

Let m_i, $1 \le i \le n$, be the multiplicity of the root of p' at x_i. By Taylor's Theorem we have

$$(*) \quad p'(x) = \frac{p^{(m_i+1)}(\gamma)}{m_i!}(x - x_i)^{m_i},$$ where γ is between x_i and x.

Since $p^{(m_i+1)}(x_i) \neq 0$, either $p^{(m_i+1)} > 0$ in a neighborhood of x_i or $p^{(m_i+1)} < 0$ in a neighborhood of x_i. Thus in order for $p'(x)$ to change sign at x_i, it follows from (*) that m_i must be odd. Similarly, if $p'(x)$ does not change sign then m_i must be even.

Therefore the number of roots of p' on (a,b), counting their multiplicities, is

$$m_1 + \dots + m_n = (\underbrace{m_1 + \dots + m_k}_{\text{odd}}) + (\underbrace{m_{k+1} + \dots + m_n}_{\text{even}})$$

$$= \text{odd}$$

28.0 Define $F(h) = f(a+h) - 2f(a) + f(a-h)$
$G(h) = h^2$.

$F(0) = G(0) = 0$, F and G are continuous for $h \geq 0$, and they are differentiable for $h > 0$. Thus by the Cauchy Mean Value Theorem (or L'Hospital's Rule)

$$\lim_{h\to 0} \frac{F(h)}{G(h)} = \lim_{h\to 0} \frac{F'(h)}{G'(h)}$$ if the latter limit exists.

But $$\lim_{h\to 0} \frac{F'(h)}{G'(h)} = \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h} = f''(a)$$

by Problem 27.L (applied with $c=a$, $x=a-h$, $y=a+h$).

29.D Let $\varepsilon > 0$. Let A be the set of rationals of the form $\frac{p}{q} \in I$, where p, q do not have common factors (except 1) and $q < \frac{1}{\varepsilon}$. We also include 0 in the set A. Suppose that A has m elements.

Let P_ε be any partition of $[0,1]$ composed of intervals whose lengths do not exceed $\frac{\varepsilon}{2m}$. Let $P_\varepsilon \subseteq P = (y_0, \dots, y_n)$.

The number of intervals $[y_{k-1}, y_k]$ such that $[y_{k-1}, y_k] \cap A \neq \phi$ is less than or equal to $2m$.

Therefore

$$|S(P;f)| = S(P;f) = \sum_{k=1}^{n} f(\zeta_k)(y_k - y_{k-1})$$

$$\leq 2m \cdot \frac{\varepsilon}{2m} + \varepsilon = 2\varepsilon.$$

Above we used the fact that if $[y_{k-1}, y_k] \cap A \neq \phi$ then $f(\zeta_k) \leq 1$, and if $[y_{k-1}, y_k] \cap A = \phi$ then $f(\zeta_k) \leq \varepsilon$.

Thus f is Riemann integrable and $\int_0^1 f = 0$.

29.G Let $|f(x)| \le M$ for all $x \in I$. Let $\varepsilon > 0$, and let $P_\varepsilon = (x_0, \dots, x_m)$ be such that if $P_\varepsilon \subseteq P$ then

$$\left| S(P;f) - \int_0^1 f \right| < \varepsilon.$$

Denote $Q = (0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1)$. Let $P = (y_0, \dots, y_N)$ be composed of points of P_ε and Q. Then $P_\varepsilon \subseteq P$. Let

$$S(Q;f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \frac{1}{n}$$

and

$$S(P;f) = \sum_{j=1}^{N} f(\zeta_j)(y_j - y_{j-1}),$$

where $\zeta_j = \frac{k}{n}$ if $y_j = \frac{k}{n}$ and $y_{j-1} = \frac{k-1}{n}$ for some $k = 1, \dots, n$, i.e. if $[y_{j-1}, y_j]$ does not contain any of the points $\{x_1, \dots, x_{m-1}\} \setminus \{\frac{k}{n} : k = 0, 1, \dots, n\}$ and $[y_{j-1}, y_j]$ is one of the subintervals of the partition Q. The sum of the lengths of intervals $[y_{j-1}, y_j]$ such that either y_{j-1} or y_j belongs to $\{x_1, \dots, x_{m-1}\} \setminus \{\frac{k}{n} : k = 0, 1, \dots, n\}$ is at most $(m-1)\frac{1}{n}$, ~~Therefore~~ and the sum of lengths of intervals $[\frac{k-1}{n}, \frac{k}{n}]$ containing them is also at most $(m-1)\frac{1}{n}$.

$$\left| S(Q;f) - \int_0^1 f \right| \le |S(Q;f) - S(P;f)| + \left| S(P;f) - \int_0^1 f \right|$$

$$\le 2\frac{m-1}{n} M + \varepsilon < 2\varepsilon \quad \text{if } n > \frac{2(m-1)M}{\varepsilon}.$$

Therefore $\lim_{n \to \infty} \left(\frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right) = \int_0^1 f$

(29.7) Let $x_1, \dots, x_m$ be the points such that $f_1(x_i) \neq f(x)$. We have

$$f_1 = f + h$$

where
$$h(x) = \begin{cases} 0 & x \in [a,b] \setminus \{x_1, \dots, x_m\} \\ f_1(x) - f(x) & x \in \{x_1, \dots, x_m\}. \end{cases}$$

Therefore it is enough to show that h is Riemann integrable and $\int_a^b h = 0$.

Let $\varepsilon > 0$ and let P_ε be composed of points $x_i \pm \varepsilon$, $i = 1, \dots, m$, which belong to $[a,b]$. If $P_\varepsilon \subseteq P = (y_0, y_1, \dots, y_n)$, we have

$$|S(P;h)| \leq \sum_{k=1}^{n} |h(\xi_k)| (y_k - y_{k-1})$$

$$\leq \left(\max_{x \in [a,b]} |h(x)| \right) m \, 2\varepsilon$$

$\therefore \int_a^b h$ exists and $\int_a^b h = 0$.

30.B Let c be a point of discontinuity of f. Then:

Case 1. There is a sequence (x_n) in $[a,b]$ such that $x_n < c$, $\lim_{n\to\infty} x_n = c$, and

$$|f(x_n) - f(c)| \geq \varepsilon_0$$

for some $\varepsilon_0 > 0$.

Case 2. There is a sequence (x_n) in $[a,b]$ such that $x_n > c$, $\lim_{n\to\infty} x_n = c$, and $|f(x_n) - f(c)| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$.

Case 1. Define

$$g(x) = \begin{cases} 0 & a \leq x < c \\ 1 & c \leq x \leq b \end{cases}$$

Let P be any partition, $P = (y_0, y_1, \ldots, y_m)$, such that $c = y_i$ is a partition point, and let $\xi_1, \ldots, \xi_m$ be any intermediate points such that $\xi_i = c$.
Let P_1 be the same partition P with the same intermediate points $\xi_1, \ldots, \xi_m$ except that now we take ξ_i to be one of the points x_n in the interval $[y_{i-1}, y_i]$. Then

$$|S(P; f, g) - S(P_1; f, g)| = |f(x_n) - f(c)| \geq \varepsilon_0.$$

Thus f is not g-integrable.

Case 2. Define

$$g(x) = \begin{cases} 0 & a \leq x \leq c \\ 1 & c < x \leq b \end{cases}$$

Similar argument shows that f is not g-integrable.

30. D Let $I = [0,1]$ and

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

f is not Riemann integrable. However $|f| = f^2 = 1$ which is Riemann integrable.